TETRAHEDRON AND 3D REFLECTION EQUATIONS FROM QUANTIZED ALGEBRA OF FUNCTIONS

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Abstract

Soibelman's theory of quantized function algebra $A_q(\mathrm{SL}_n)$ provides a representation theoretical scheme to construct a solution of the Zamolodchikov tetrahedron equation. We extend this idea originally due to Kapranov and Voevodsky to $A_q(\mathrm{Sp}_{2n})$ and obtain the intertwiner K corresponding to the quartic Coxeter relation. Together with the previously known 3-dimensional (3D) R matrix, the K yields the first ever solution to the 3D analogue of the reflection equation proposed by Isaev and Kulish. It is shown that matrix elements of R and K are polynomials in q and that there are combinatorial and birational counterparts for R and K. The combinatorial ones arise either at $q = 0$ or by tropicalization of the birational ones. A conjectural description for the type B and F_4 cases is also given.

1. Introduction

The tetrahedron equation [43, 44] is the 3-dimensional (3D) analogue of the Yang-Baxter equation [2]. It has been well recognized by now that quantum groups [11, 15] provide a comprehensive framework for the algebraic aspect of the latter. The tetrahedron equation is more challenging but many efforts and results continue to emerge until today. See [3, 23, 32, 5, 30, 21, 24, 12, 18, 25, 37, 19, 36, 35, 20, 7, 6, 29] for example. In this paper we study the tetrahedron equation and its generalizations based on representation theory of quantized algebra of functions. Let us briefly review this approach which is seemingly long forgotten, to motivate our work.

Let $\mathfrak{g}$ be a classical simple Lie algebra and G be the corresponding Lie group. The quantized algebra of functions on G or q deformation of the coordinate ring of G is the Hopf algebra dual to the quantized universal enveloping algebra $U_q(\mathfrak{g})$. We denote it by $A_q(G)$ in this paper and assume that q is generic unless otherwise stated. It has been studied from a variety of viewpoints. See [11, 34, 33, 41, 39, 40] for example. The simplest one is $A_q(\mathrm{SL}_2) = \langle t_{11}, t_{12}, t_{21}, t_{22}\rangle$ with the relations

$$\begin{aligned} &t_{11}t_{21} = qt_{21}t_{11}, \quad t_{12}t_{22} = qt_{22}t_{12}, \quad t_{11}t_{12} = qt_{12}t_{11}, \quad t_{21}t_{22} = qt_{22}t_{21}, \\ &[t_{12}, t_{21}] = 0, \quad [t_{11}, t_{22}] = (q - q^{-1})t_{21}t_{12}, \quad t_{11}t_{22} - qt_{12}t_{21} = 1. \end{aligned} \tag{1.1}$$

It has the irreducible representation in terms of the q-oscillator acting on the Fock space [41]. See (2.4)–(2.6). Irreducible representations of $A_q(G)$ for general G were classified by Soibelman [39, 40]. Associated with each vertex i of the Dynkin diagram, $A_q(G)$ has an irreducible representation π_i which factors through the projection to $A_q(\mathrm{SL}_2)$ corresponding to i. The representations $\pi_1, \ldots, \pi_n$ ($n = \mathrm{rank}\, G$) play the role of fundamental representations. General irreducible representations are in one to one correspondence with elements of the Weyl group $W(G) = \langle s_1, \ldots, s_n\rangle$ up to some "torus degrees of freedom". More concretely if $w = s_{i_1} \cdots s_{i_r} \in W(G)$ is a reduced expression by the simple reflections, the corresponding irreducible $A_q(G)$ module is realized as the tensor product of the fundamental ones as $\pi_{i_1} \otimes \cdots \otimes \pi_{i_r}$. A crucial consequence of this claim is the equivalence $\pi_{i_1} \otimes \cdots \otimes \pi_{i_r} \simeq \pi_{j_1} \otimes \cdots \otimes \pi_{j_r}$ for arbitrary

reduced expressions $w = s_{j_1} \cdots s_{j_r}$. In particular it ensures the existence of a unique (up to normalization) isomorphism between such tensor products, which we call the *intertwiner*.

In [18] Kapranov and Voevodsky found an application of these results for type A to the tetrahedron equation. The Coxeter relations $s_2s_1s_2 = s_1s_2s_1$ and $s_3s_2s_3 = s_2s_3s_2$ in $W(\mathrm{SL}_4)$ imply the equivalence of the irreducible $A_q(\mathrm{SL}_4)$ modules $\pi_2 \otimes \pi_1 \otimes \pi_2 \simeq \pi_1 \otimes \pi_2 \otimes \pi_1$ and $\pi_3 \otimes \pi_2 \otimes \pi_3 \simeq \pi_2 \otimes \pi_3 \otimes \pi_2$, therefore the existence of unique (up to normalization) intertwiners $\Phi^{(i)}$ satisfying

$$\begin{aligned}(\pi_2 \otimes \pi_1 \otimes \pi_2) \circ \Phi^{(1)} &= \Phi^{(1)} \circ (\pi_1 \otimes \pi_2 \otimes \pi_1),\\ (\pi_3 \otimes \pi_2 \otimes \pi_3) \circ \Phi^{(2)} &= \Phi^{(2)} \circ (\pi_2 \otimes \pi_3 \otimes \pi_2).\end{aligned} \tag{1.2}$$

On the other hand the two reduced expressions of the longest element $s_1s_2s_3s_1s_2s_1 = s_3s_2s_3s_1s_2s_3$ lead to the equivalence $\pi_1 \otimes \pi_2 \otimes \pi_3 \otimes \pi_1 \otimes \pi_2 \otimes \pi_1 \simeq \pi_3 \otimes \pi_2 \otimes \pi_3 \otimes \pi_1 \otimes \pi_2 \otimes \pi_3$. Up to transposition of components, the intertwiner for this can be constructed as the composition of the form $\Phi^{(2)}\Phi^{(1)}\Phi^{(2)}\Phi^{(1)}$ in two ways, which parallel the transformations of the reduced expressions using the Coxeter relations. See (2.37) for the precise description. Since the intertwiner is unique, it enforces the consistency condition of the form $\Phi^{(2)}\Phi^{(1)}\Phi^{(2)}\Phi^{(1)} = \Phi^{(2)}\Phi^{(1)}\Phi^{(2)}\Phi^{(1)}$ up to transposition operators (cf. (2.38)). It turns out that $\Phi^{(1)}$ and $\Phi^{(2)}$ yield essentially the same matrix acting on the tensor cube of the Fock space. Regarding them as the 3D R matrix, the consistency condition is nothing but the tetrahedron equation.

The discovery by Kapranov and Voevodsky was rephrased in [22], but did not seem to have plentiful citations in the mathematical physics community working on the tetrahedron equation, possibly due to some unfortunate misprints. However in retrospect, it was offering a proper quantum group theoretical framework for the ideas being developed independently at about the same time in several guises called local Yang-Baxter equation, tetrahedral Zamolodchikov algebra and quantum Korepanov equation, etc [32, 24, 25, 20]. In particular the 3D R matrix obtained relatively recently by Bazhanov, Sergeev and Mangazeev [7, 6] can be identified with the intertwiner that follows from the Kapranov-Voevodsky approach. See (2.29). The essence of the strategies in the works [32, 24, 25, 20, 7, 6] may roughly be stated as upgrading the *equality* in the Yang-Baxter relation to a *transformation* φ of some triple product

$$S_1S_2S_1 = S_2S_1S_2 \quad \Longrightarrow \quad \varphi : S_1S_2S_1 \longmapsto S_2'S_1'S_2' \tag{1.3}$$

and to produce a solution of the tetrahedron equation from φ. A similar idea was considered as "vectorization of triangle equations" in [18, sec.6.9]. The intertwining relation (1.2) is a realization of it as $\varphi = \mathrm{Ad}(\Phi^{(i)})$. From this viewpoint the prescription (1.3) is traced back to the transformations of reduced expressions of the Weyl group elements by means of the cubic Coxeter relation. The 3D R matrix and the conventional 2D quantum R matrices are thus put on a parallel footing. They are pinned as the intertwiners for A_q modules and U_q modules, respectively. The representations of A_q do not have the variety like the highest weight modules for U_q. However their intertwiners are known to generate a broad class of 2D quantum R matrices in some cases [7, 29].

Weyl group	Factorized scattering in 3D
reduced expressions	multi-string states
cubic Coxeter relation	3D R (scattering amplitude)
the longest element	tetrahedron equation

The story in type A digested so far naturally motivates us to explore the general G case. It may also be viewed as the 3D analogue of Cherednik's generalization of the Yang-Baxter equation along the classical Coxeter systems [10]. Our aim in this paper is to launch such a result for those G having a double arrow in the Dynkin diagrams mainly along type C, i.e. $G = \mathrm{Sp}_{2n}$. We introduce the quantized algebra of functions $A_q(\mathrm{Sp}_{2n})$ following [34]. The $n = 3$

case already covers the generic situation on which we shall focus. Irreducible representations $\pi_i (i = 1, 2, 3)$ are presented in terms of the q_i-deformed oscillators, where $(q_1, q_2, q_3) = (q, q, q^2)$ reflecting the squared lengths of the three simple roots of Sp_6. The Coxeter relations in the Weyl group $W(\mathrm{Sp}_6) = \langle s_1, s_2, s_3\rangle$ include $s_1s_2s_1 = s_2s_1s_2$ and $s_2s_3s_2s_3 = s_3s_2s_3s_2$. Accordingly we have the equivalence of the tensor product representations

$$\pi_1 \otimes \pi_2 \otimes \pi_1 \simeq \pi_2 \otimes \pi_1 \otimes \pi_2, \qquad \pi_2 \otimes \pi_3 \otimes \pi_2 \otimes \pi_3 \simeq \pi_3 \otimes \pi_2 \otimes \pi_3 \otimes \pi_2. \tag{1.4}$$

The problem is to construct their intertwiners explicitly and formulate a generalization of the tetrahedron equation that should emerge as a consistency condition among them. We find that the former in (1.4) leads to essentially the same intertwiner R (or Φ as in (2.14)) as type A. The intertwiner for the latter is a new object, which will be denoted by K (or Ψ as in (3.21)). We present its explicit formula in Theorem 3.4. The R and K act on tensor cube and tensor quartet of the Fock space, respectively. However they are locally finite with respect to the natural basis of the Fock space, i.e. they are decomposed into direct sums of finite dimensional matrices specified by conserved quantities. The consistency condition involving R and K is derived from the longest element of $W(\mathrm{Sp}_6)$. The result takes the form $RRKRRKK = KKRRKRR$. See (3.35) and (3.36). We call it the 3D reflection equation as it is a natural 3D analogue of the 2D reflection equation going back to [10, 38, 26]. The physical meaning of it is a factorization condition of 3 strings scattering in 3D with boundary reflections, where the R and K stand for the amplitudes of 3 string scattering and reflection at the boundary respectively. Such a 3D system was originally considered by Isaev and Kulish [14] who proposed the "tetrahedron reflection equation". It turns out that our 3D reflection equation coincide exactly with the constant version (spectral parameter-free case) of their tetrahedron reflection equation[1]. Our Theorem 3.4 yields the first ever solution to it.

There is another persistent theme in this paper. We have seen that the R and K are intertwiners corresponding to the Coxeter relations $(s_is_j)^{m_{ij}} = 1$ with $m_{ij} = 3$ and $m_{ij} = 4$, respectively. They are *quantum* objects controlled by the quantum algebra $A_q(G)$. We shall establish that the both R and K possess *polynomial* matrix elements in q and form a triad with their *birational* and *combinatorial* counterparts. They all satisfy tetrahedron and 3D reflection equations in the respective setting. The combinatorial ones are bijections among finite sets, which arise either at $q = 0$ of the quantum R and K or by the *tropicalization (ultradiscretization)* of the birational R and K. These features are quite analogous to the quantum R matrices in 2D (cf. [27, 13]) and have been summarized in Table 1. The birational R and K are the maps (2.42) and (3.40) characterized by the identities (2.43), (3.38) and (3.39) among the generators $G_i(x), X_i(x), Y_i(x)$ of the unipotent subgroup of G. These matrix identities are related to the independence of Schubert cells X_w for Weyl group elements w on the reduced expressions. Soibelman's theory on $A_q(G)$ forms a quantum analogue of this aspect as stressed in [39, 40]. We note that the combinatorial 3D R has effectively appeared already in [31]. The birational 3D R has also been encountered in various contexts. See for example [8, 42] and [35, 20], where in the latter reference the tetrahedron equation for them has been called functional tetrahedron equation.

	quantum	combinatorial	birational
3D R	$\mathcal{R}$ (2.20)	$\mathscr{R}$ (2.32)	$\mathbf{R}$ (2.42)
3D K	$\mathcal{K}$ (Th.3.4)	$\mathscr{K}$ (Th.3.5)	$\mathbf{K}$ (3.40)

TABLE 1. Triad of 3D R and 3D K

The layout of the paper is as follows. In Section 2 we begin by reviewing the type A case following the idea of the paper [18] in 1994. See also [22]. It may be viewed as a formulation

[1] The authors are indebted to A. Isaev for a comment on this point.

of many parallel ideas and results in [24, 25, 20, 7, 6] in terms of the representation theory of $A_q(\mathrm{SL}_n)$ [41, 39, 40]. We point out in (2.29) the coincidence of the intertwiner in [18] (up to misprints) and the 3D R in [7, 6]. Another observation which appears new is the polynomiality of the 3D R in q (Remark 2.6), which leads to the combinatorial 3D R. The triad of quantum, birational and combinatorial R's are formulated in a unified perspective and summarized in Table 1.

In Section 3 we deal with type C case. We obtain the intertwiner K corresponding to the quartic Coxeter relation and show that it also forms the triad with the birational and combinatorial counterparts. We formulate the 3D reflection equation involving R and K and argue the relation with the physical setting of the tetrahedron reflection equation by Isaev and Kulish [14].

In Section 4 we present the intertwiners and their consistency conditions for type B and F_4 cases on conjectural basis. They are natural candidates indicated from the results on type C. The type B case yields the second (conjectural) solution of the same 3D reflection equation.

Appendix A gives the list of intertwining relations for K. Appendices B and C contain the technical details of the proofs of Theorem 3.4 and Theorem 3.5.

2. SL case

2.1. Quantized algebra of functions $A_q(\mathrm{SL}_n)$. We begin by recalling the quantized algebra of functions of type A. It has been studied and denoted in many ways as $\mathrm{Fun}(\mathrm{SL}_q(n))$ [34], $\mathbb{C}[G]_h$ [41], $A(SL_q(n;\mathbb{C}))$ [33], $\mathbb{C}[SU(2)]_q$ ($n=2$ case) [40], $\mathbb{C}[GL(n)_q]$ (GL case rather than SL) [18], $\mathbb{C}[SL_2(\mathbb{C})](q)$ [22] and so on. In this paper we write it as $A_q(\mathrm{SL}_n)$. The $A_q(\mathrm{SL}_n)$ is a Hopf algebra [1] generated by $T=(t_{ij})_{1\le i,j\le n}$ with relations. They are presented in the so called $RTT=TTR$ form and the unit quantum determinant condition:

$$\sum_{m,p} R_{ij,mp}t_{mk}t_{pl} = \sum_{m,p} t_{jp}t_{im}R_{mp,kl}, \tag{2.1}$$

$$\sum_{\sigma\in\mathfrak{S}_n} (-q)^{l(\sigma)}t_{1\sigma_1}\cdots t_{n\sigma_n} = 1. \tag{2.2}$$

Here $l(\sigma)$ denotes the length of the permutation σ and the structure constant is specified by

$$\sum_{i,j,k,l} R_{ij,kl}E_{ik}\otimes E_{jl} = q\sum_i E_{ii}\otimes E_{ii} + \sum_{i\neq j}E_{ii}\otimes E_{jj} + (q-q^{-1})\sum_{i>j}E_{ij}\otimes E_{ji},$$

where the indices are summed over $1,2,\ldots,n$, and E_{ij} is a matrix unit. This matrix is extracted as

$$\sum_{i,j,m,l} R_{ij,ml}E_{im}\otimes E_{jl} = q\lim_{x\to\infty} x^{-1}R(x)|_{k=q^{-1}}$$

from the quantum R matrix $R(x)$ [4, 16] for the vector representation of $U_q(A^{(1)}_{n-1})$ given in [16, eq.(3.5)]. Explicitly, the relation (2.1) reads as (see for example [33])

$$[t_{ik},t_{jl}] = \begin{cases} 0 & (i<j, k>l),\\ (q-q^{-1})t_{jk}t_{il} & (i<j,k<l),\end{cases}$$
$$t_{ik}t_{jk} = qt_{jk}t_{ik}\ (i<j), \quad t_{ki}t_{kj} = qt_{kj}t_{ki}\ (i<j).$$

The case $n=2$ is given by (1.1). The coproduct is the standard one:

$$\Delta(t_{ij}) = \sum_k t_{ik}\otimes t_{kj}. \tag{2.3}$$

We will use the same symbol Δ flexibly to also mean the multiple coproducts like $(\Delta\otimes 1)\circ\Delta = (1\otimes\Delta)\circ\Delta$, etc. We omit the antipode and the counit for they will not be used in this paper.

2.2. **Representations π_i.** Let $\mathrm{Osc}_q = \langle \mathbf{1}, \mathbf{a}^+, \mathbf{a}^-, \mathbf{k}\rangle$ be the q-oscillator algebra, i.e. an associative algebra with the relations

$$\begin{aligned}&\mathbf{k}\,\mathbf{a}^+ = q\,\mathbf{a}^+\mathbf{k}, \quad \mathbf{k}\,\mathbf{a}^- = q^{-1}\mathbf{a}^-\mathbf{k},\\ &\mathbf{a}^-\mathbf{a}^+ = \mathbf{1} - q^2\mathbf{k}^2, \quad \mathbf{a}^+\mathbf{a}^- = \mathbf{1} - \mathbf{k}^2, \quad [\mathbf{1}, \text{everything}] = 0.\end{aligned} \tag{2.4}$$

It has a representation on the Fock space $\mathcal{F}_q = \oplus_{m\ge 0}\mathbb{C}(q)|m\rangle$:

$$\mathbf{1}|m\rangle = |m\rangle,\ \mathbf{k}|m\rangle = q^m|m\rangle,\ \mathbf{a}^+|m\rangle = |m+1\rangle,\ \mathbf{a}^-|m\rangle = (1-q^{2m})|m-1\rangle. \tag{2.5}$$

The label m will be referred as occupation number of the q-oscillator.

Consider the maps $\pi_i\,(1 \le i \le n-1) : A_q(\mathrm{SL}_n) \to \mathrm{Osc}_q$ that send the generators $T = (t_{ij})$ as

$$\begin{pmatrix} t_{11} & & & & & & t_{1n}\\ & \ddots & & & & & \\ & & t_{i-1,i-1} & & & & \\ & & & t_{i,i} & t_{i,i+1} & & \\ & & & t_{i+1,i} & t_{i+1,i+1} & & \\ & & & & & t_{i+2,i+2} & \\ & & & & & & \ddots \\ t_{n1} & & & & & & t_{nn}\end{pmatrix} \mapsto \begin{pmatrix} \sigma_i\mathbf{1} & & & & & & \\ & \ddots & & & & & \\ & & \sigma_i\mathbf{1} & & & & \\ & & & \mu_i\mathbf{a}^- & \alpha_i\mathbf{k} & & \\ & & & \beta_i\mathbf{k} & \nu_i\mathbf{a}^+ & & \\ & & & & & \sigma_i\mathbf{1} & \\ & & & & & & \ddots \\ & & & & & & & \sigma_i\mathbf{1}\end{pmatrix}, \tag{2.6}$$

where all the blanks on the RHS mean 0. $\alpha_i, \beta_i, \mu_i, \nu_i$ and σ_i are parameters obeying the relation

$$\alpha_i\beta_i = -q\sigma_i^{-n+2}, \quad \mu_i\nu_i = \sigma_i^{-n+2}. \tag{2.7}$$

It is elementary to show

Proposition 2.1. *The maps $\pi_1, \dots, \pi_{n-1}$ are naturally extended to the algebra homomorphisms. The resulting representations of $A_q(\mathrm{SL}_n)$ on $\mathcal{F}_q$ are irreducible.*

Let

$$P(x\otimes y) = y\otimes x \tag{2.8}$$

be the transposition, where x and y are taken from Osc_q (or its representation $\mathrm{End}(\mathcal{F}_q)$). For $|i-j|\ge 2$, one can check $P(\pi_i\otimes\pi_j)(\Delta(f)) = (\pi_j\otimes\pi_i)(\Delta(f))P$ for any $f\in A_q(\mathrm{SL}_n)$. This is due to the $f = t_{km}$ case

$$P\Big(\sum_l \pi_i(t_{kl})\otimes\pi_j(t_{lm})\Big) = \Big(\sum_l \pi_j(t_{kl})\otimes\pi_i(t_{lm})\Big)P \quad \text{for}\ \ |i-j|\ge 2 \tag{2.9}$$

for any k and m. The point here is that the naively obtained expression $(\sum_l \pi_j(t_{lm})\otimes\pi_i(t_{kl}))P$ violates the coproduct structure (2.3), but it equals to the RHS of (2.9) thanks to the simple structure of the matrix (2.6).

In what follows, we set $\sigma_i = 1$ for all i, which does not cause an essential loss of generality. See Remark 2.5. The representations $A_q(\mathrm{SL}_n)\to\mathrm{End}(\mathcal{F}_q)$ defined by (2.5), (2.6) with

$$\alpha_i\beta_i = -q, \quad \mu_i\nu_i = 1 \tag{2.10}$$

will also be denoted by $\pi_i = \pi_i^{\alpha_i,\mu_i}$.

Let $W(\mathrm{SL}_n) = \langle s_1, \dots, s_{n-1}\rangle$ be the Weyl group of SL_n. It is a Coxeter system with generators $s_1, \dots, s_{n-1}$ obeying the relations

$$s_i^2 = 1, \qquad s_is_j = s_js_i\ \ (|i-j|\ge 2), \qquad s_is_js_i = s_js_is_j\ \ (|i-j| = 1).$$

We will often abbreviate $\pi_{i_1}\otimes\pi_{i_2}\otimes\cdots\otimes\pi_{i_r}$ to $\pi_{i_1,i_2,\dots,i_r}$ in the sequel. These indices should not be confused with the ones appearing later signifying the *positions* in the multiple tensor products.

Theorem 2.2 ([39, 40])**.** *(i) The $A_q(\mathrm{SL}_n)$-module $\pi_{i_1,i_2,\ldots,i_r}$ is irreducible if $s_{i_1}s_{i_2}\cdots s_{i_r}$ is a reduced expression of an element from $W(\mathrm{SL}_n)$.*

(ii) If $s_{i_1}s_{i_2}\cdots s_{i_r} = s_{j_1}s_{j_2}\cdots s_{j_r}$ are two reduced expressions, then the two irreducible representations $\pi_{i_1,i_2,\ldots,i_r}$ and $\pi_{j_1,j_2,\ldots,j_r}$ are equivalent.

The same theorem holds also for $A_q(G)$ for any simple Lie group G, where π_i is associated to each node i of the Dynkin diagram [39, 40].

2.3. $\boldsymbol{A_q(\mathrm{SL}_3)}$ **and intertwiner.** The isomorphism of the two irreducible representations will be called *intertwiner*. By Schur's lemma, it is unique up to an overall normalization. The first nontrivial situation arises in $A_q(\mathrm{SL}_3)$, where one has the equivalence $\pi_{121} \simeq \pi_{212}$ reflecting the Coxeter relation $s_1s_2s_1 = s_2s_1s_2$. Let

$$\Phi : \mathcal{F}_q \otimes \mathcal{F}_q \otimes \mathcal{F}_q \longrightarrow \mathcal{F}_q \otimes \mathcal{F}_q \otimes \mathcal{F}_q \tag{2.11}$$

be the associated intertwiner. It is characterized by the relations:

$$\pi_{212}(\Delta(f)) \circ \Phi = \Phi \circ \pi_{121}(\Delta(f)) \quad (\forall f \in A_q(\mathrm{SL}_3)), \tag{2.12}$$

$$\Phi(|0\rangle \otimes |0\rangle \otimes |0\rangle) = |0\rangle \otimes |0\rangle \otimes |0\rangle, \tag{2.13}$$

where the latter just fixes a normalization. As in the case of quantum R matrices, we find it convenient to work with R defined by

$$R = \Phi P_{13} : \ \mathcal{F}_q \otimes \mathcal{F}_q \otimes \mathcal{F}_q \longrightarrow \mathcal{F}_q \otimes \mathcal{F}_q \otimes \mathcal{F}_q, \tag{2.14}$$

where $P_{13}(x \otimes y \otimes z) = z \otimes y \otimes x$. The intertwining relation for R reads

$$\pi_{212}(\Delta(f)) \circ R = R \circ \pi_{121}(\tilde{\Delta}(f)) \quad (\forall f \in A_q(\mathrm{SL}_3)), \tag{2.15}$$

where $\tilde{\Delta}(f) = P_{13}(\Delta(f))P_{13}$, namely,

$$\tilde{\Delta}(t_{ij}) = \sum_{l_1,l_2} t_{l_2 j} \otimes t_{l_1 l_2} \otimes t_{i l_1}. \tag{2.16}$$

For instance, the equation (2.15) with $f = t_{11}$ gives

$$\mu_1(\mathbf{1} \otimes \mathbf{a}^- \otimes \mathbf{1})R = R\big(\mu_1^2(\mathbf{a}^- \otimes \mathbf{1} \otimes \mathbf{a}^-) - q\mu_2(\mathbf{k} \otimes \mathbf{a}^- \otimes \mathbf{k})\big). \tag{2.17}$$

The R is regarded as a matrix $R = (R^{abc}_{ijk})$ whose elements are specified by

$$R(|i\rangle \otimes |j\rangle \otimes |k\rangle) = \sum_{a,b,c} R^{abc}_{ijk} |a\rangle \otimes |b\rangle \otimes |c\rangle. \tag{2.18}$$

The normalization condition (2.13) becomes $R^{000}_{000} = 1$. Introduce the notations

$$(q)_i = \prod_{j=1}^{i}(1-q^j), \quad \left\{ \begin{matrix} i_1, \ldots, i_r \\ j_1, \ldots, j_s \end{matrix} \right\} = \begin{cases} \frac{\prod_{k=1}^{r}(q^2)_{i_k}}{\prod_{k=1}^{s}(q^2)_{j_k}} & \forall i_k, j_k \in \mathbb{Z}_{\geq 0}, \\ 0 & \text{otherwise}. \end{cases}$$

Except the dependence on the parameters μ_i, the following is essentially due to [18].

Theorem 2.3. *The equation (2.15) with $R^{000}_{000} = 1$ has a unique solution. It has the form*

$$R^{abc}_{ijk} = \mu_1^{a-j+k}\mu_2^{b-a-k}\mathscr{R}^{abc}_{ijk}, \tag{2.19}$$

where $\mathscr{R}^{abc}_{ijk}$ is independent of the parameters and given by

$$\mathscr{R}^{abc}_{ijk} = \delta_{i+j,a+b}\delta_{j+k,b+c} \sum_{\lambda+\mu=b} (-1)^{\lambda} q^{i(c-j)+(k+1)\lambda+\mu(\mu-k)} \left\{ \begin{matrix} i, j, c+\mu \\ \mu, \lambda, i-\mu, j-\lambda, c \end{matrix} \right\}, \tag{2.20}$$

where the sum is over $\lambda, \mu \in \mathbb{Z}_{\geq 0}$ such that $\lambda + \mu = b$.

Proof. For the choices $f = t_{13}$ and t_{31}, (2.15) reads

$$(q^{b+c} - q^{j+k})R^{abc}_{ijk} = 0, \quad (q^{a+b} - q^{i+j})R^{abc}_{ijk} = 0$$

giving the factor $\delta_{i+j,a+b}\delta_{j+k,b+c}$ in (2.20). We call this support property *conservation law.* Thanks to it, one readily checks that $\mathcal{R}^{abc}_{ijk}$ defined in (2.19) obeys the recursion relations independent of the parameters. Let us pick the following two among them:

$$f = t_{32} : \mathcal{R}^{abc}_{ijk} = (1 - q^{2c+2})q^{a-j}\mathcal{R}^{a,b-1,c+1}_{i-1,j,k} + q^{c-j}\mathcal{R}^{a-1,b,c}_{i-1,j,k}, \tag{2.21}$$

$$f = t_{33} : \mathcal{R}^{abc}_{ijk} = \mathcal{R}^{a-1,b,c-1}_{i,j-1,k} - q^{a+c+1}\mathcal{R}^{a,b-1,c}_{i,j-1,k}. \tag{2.22}$$

They can be iterated m times to reduce i and j indices as

$$\mathcal{R}^{abc}_{ijk} = \delta_{i+j,a+b}\delta_{j+k,b+c}\sum_{r=0}^{m} q^{(m-r)(c-j)+r(a-j-m+r)} \begin{Bmatrix} m, c+r \\ r, m-r, c \end{Bmatrix} \mathcal{R}^{a-m+r,b-r,c+r}_{i-m,j,k},$$

$$\mathcal{R}^{abc}_{ijk} = \delta_{i+j,a+b}\delta_{j+k,b+c}\sum_{r=0}^{m} (-1)^r q^{r(a+c-2m+2r+1)} \begin{Bmatrix} m \\ r, m-r \end{Bmatrix} \mathcal{R}^{a-m+r,b-r,c-m+r}_{i,j-m,k}.$$

Combining them, we get (2.20). Since the intertwiner exists, the validity of (2.15) for the other f's is guaranteed. □

Proposition 2.4. *The matrices R and $\mathcal{R} = (\mathcal{R}^{abc}_{ijk})$ have the properties*

$$R^{-1} = R|_{\mu_1 \leftrightarrow \mu_2}, \qquad \mathcal{R}^{-1} = \mathcal{R}, \tag{2.23}$$

$$(q^2)_a(q^2)_b(q^2)_c\,\mathcal{R}^{abc}_{ijk} = (q^2)_i(q^2)_j(q^2)_k\,\mathcal{R}^{ijk}_{abc}. \tag{2.24}$$

Proof. Take $\alpha_i = \beta_i$ in (2.6) without influencing the relations (2.10), (2.19) and the parameters μ_i and ν_i. Then $\pi_\ell(t_{ij}) = \pi_\ell(t_{ji})$ holds for any i, j, ℓ, and therefore $\pi_{121}(\tilde{\Delta}(t_{ij})) = \pi_{121}(\Delta(t_{ji}))$ by (2.16). Thus the defining equations (2.15) for R are equivalent to those for R^{-1} with the interchange $\mu_1 \leftrightarrow \mu_2$. This proves the first equality in (2.23), which also implies the second one by (2.19). Next to show (2.24) we tune the parameter as $\mu_1 = \mu_2 = \nu_1 = \nu_2 = 1$, $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$ without violating (2.10). Then we have $R = \mathcal{R}$ by (2.19), hence (2.24) is the assertion that $\mathcal{D}R$ is symmetric. Here $\mathcal{D} = D \otimes D \otimes D$ with $D \in \mathrm{End}(\mathcal{F}_q)$ being a diagonal operator defined by

$$D|m\rangle = (q^2)_m|m\rangle. \tag{2.25}$$

From (2.5) we see $(\mathbf{a}^{\pm})^T = D\mathbf{a}^{\mp}D^{-1}$ and $(\mathbf{k})^T = D\mathbf{k}D^{-1}$ for the transposed actions on the Fock space. Under the above choice of the parameters, this leads to

$$\pi_1(t_{ij})^T = D\pi_2(t_{j'i'})D^{-1}, \quad \pi_2(t_{ij})^T = D\pi_1(t_{j'i'})D^{-1} \quad (i' = 4 - i)$$

for the single representations (2.6), therefore

$$(\pi_{212}\Delta(t_{ij}))^T = \mathcal{D}(\pi_{121}\tilde{\Delta}(t_{j'i'}))\mathcal{D}^{-1}, \quad (\pi_{121}\tilde{\Delta}(t_{ij}))^T = \mathcal{D}(\pi_{212}\Delta(t_{j'i'}))\mathcal{D}^{-1}$$

for their tensor products. See (2.16). Taking the transpose of (2.15) by means of the above formula, one finds that $\mathcal{D}^{-1}R^T\mathcal{D}$ again satisfies (2.15). Since the intertwiner is unique up to normalization and R and $\mathcal{D}^{-1}R^T\mathcal{D}$ coincide on $|0\rangle \otimes |0\rangle \otimes |0\rangle$, we conclude $\mathcal{D}^{-1}R^T\mathcal{D} = R$ hence $(\mathcal{D}R)^T = \mathcal{D}R$. □

Remark 2.5. If σ_1 and σ_2 are retained in (2.6), the intertwining relation (2.15) has the solution only if $\sigma_1 = \sigma_2$. The resulting modification of (2.19) is only to multiply an overall power of σ_1 on its RHS.

The equation (2.12) or (2.15) have also been considered effectively in several guises and referred as tetrahedral Zamolodchikov algebra, local Yang-Baxter equation or quantum Korepanov equation, etc. See for example [32, 24, 25, 20, 7, 6].

The result (2.20) has been obtained by using (2.15) with $f = t_{32}$ and t_{33}. It is the same route as those taken in [18] and [22], although the formulas therein contain misprints unfortunately.

One can derive apparently different expressions from other choices of f. Here we include a remark on the choice $f = t_{11}$ given in (2.17). In terms of the matrix elements of $\mathscr{R}$ it reads

$$(1-q^{2b})\mathscr{R}^{abc}_{ijk} = (1-q^{2i})(1-q^{2k})\mathscr{R}^{a,b-1,c}_{i-1,j,k-1} - q^{i+k+1}(1-q^{2j})\mathscr{R}^{a,b-1,c}_{i,j-1,k}. \tag{2.26}$$

On the other hand, recall the 3D R matrix [7, eq.(30)] whose elements are given by

$$\langle i,j,k|\mathbf{r}|a,b,c\rangle = \delta_{i+j,a+b}\delta_{j+k,b+c}\frac{q^{(a-j)(c-j)}}{(q^2)_b}P_b(q^{2i},q^{2j},q^{2k}), \tag{2.27}$$

where P_m is determined by the recursion

$$P_m(x,y,z) = (1-x)(1-z)P_{m-1}(q^{-2}x,y,q^{-2}z) - q^{2-2m}xz(1-y)P_{m-1}(x,q^{-2}y,z) \tag{2.28}$$

and the initial condition $P_0(x,y,z) = 1$. In (2.27) we have removed the power $q^{-\beta}$ and the sign $(-1)^\beta$ in [7, eq.(30)] in view of the fact that the first one is absent in [6, eq.(59)] and the latter can be absorbed into ε in [6, eq.(22)]. We point out that

$$\mathscr{R}^{abc}_{ijk} = \langle i,j,k|\mathbf{r}|a,b,c\rangle. \tag{2.29}$$

To show this, substitute the RHS of (2.27) into (2.26). Then it agrees with (2.28) due to the conservation law $a+b=i+j$ and $b+c=j+k$. It remains to check the initial condition

$$\mathscr{R}^{a0c}_{ijk} = \delta_{i+j,a}\delta_{j+k,c}\, q^{(a-j)(c-j)} = \delta_{i+j,a}\delta_{j+k,c}\, q^{ik}, \tag{2.30}$$

which is straightforward by (2.20). Thus (2.29) has been proved. Note that the formula (2.27) with (2.19) tells

$$\mathscr{R}^{abc}_{ijk} = \mathscr{R}^{cba}_{kji}, \qquad R^{abc}_{ijk} = R^{cba}_{kji}. \tag{2.31}$$

Remark 2.6. One can show that $\mathscr{R}^{abc}_{ijk}$ is a *polynomial* in q with integer coefficients. More precisely, $\mathscr{R}^{abc}_{ijk} \in q^\xi\mathbb{Z}[q^2]$ where $\xi = 0,1$ is specified by $\xi \equiv (a-j)(c-j) \mod 2$. The matrix

$$\mathcal{R} = (\mathcal{R}^{abc}_{ijk}) := \mathscr{R}|_{q=0}$$

has the elements

$$\mathcal{R}^{abc}_{ijk} = \mathscr{R}^{abc}_{ijk}|_{q=0} = \delta_{i+j,a+b}\delta_{j+k,b+c}\delta_{i,b+(a-c)_+}\delta_{j,\min(a,c)}\delta_{k,b+(c-a)_+}, \tag{2.32}$$

where $(y)_+ = \max(y,0)$. The proof is far simpler than the analogous result on K in Theorem 3.5 which will be detailed in Appendix C. Moreover, (2.23) implies $\mathcal{R} = \mathcal{R}^{-1}$. Thus $\mathcal{R}$ defines a bijection on each finite set $\{(a,b,c) \in (\mathbb{Z}_{\ge 0})^3\,|a+c = \text{const}, b+c = \text{const}\}$ characterized by the values of conserved quantities. We call $\mathcal{R}$ *combinatorial* 3D R. It is analogous to the $q=0$ case of quantum R matrices, which has led to many applications. See for example [27, 13]. More remarks are in order in Section 2.5.

Example 2.7. The following is the list of all the nonzero $\mathscr{R}^{abc}_{314}$.

$$\begin{aligned}
\mathscr{R}^{041}_{314} &= -q^2(1-q^4)(1-q^6)(1-q^8),\\
\mathscr{R}^{132}_{314} &= (1-q^6)(1-q^8)(1-q^4-q^6-q^8-q^{10}),\\
\mathscr{R}^{223}_{314} &= q^2(1+q^2)(1+q^4)(1-q^6)(1-q^6-q^{10}),\\
\mathscr{R}^{314}_{314} &= q^6(1+q^2+q^4-q^8-q^{10}-q^{12}-q^{14}),\\
\mathscr{R}^{405}_{314} &= q^{12}.
\end{aligned}$$

Thus $\mathcal{R}^{abc}_{314} = \delta_{a,1}\delta_{b,3}\delta_{c,2}$ in agreement with (2.32).

2.4. **$A_q(\mathrm{SL}_4)$ and tetrahedron equation.** Consider $A_q(\mathrm{SL}_4)$ and let $\pi_i = \pi_i^{\alpha_i,\mu_i}$ $(i = 1, 2, 3)$ be its irreducible representations specified around (2.10) following [18]. The tensor products π_{212} and π_{121} are intertwined by the same Φ as the one for $A_q(\mathrm{SL}_3)$ given in (2.14) and Theorem 2.3. Write it as $\Phi^{(1)}$, which involves the parameters μ_1 and μ_2. It is easy to see that π_{323} and π_{232} is similarly intertwined by $\Phi^{(2)}$ obtained from $\Phi^{(1)}$ by changing the parameters (μ_1, μ_2) to (μ_2, μ_3). Thus we have

$$\begin{aligned} \pi_{212}(\Delta(f)) \circ \Phi^{(1)} &= \Phi^{(1)} \circ \pi_{121}(\Delta(f)), \\ \pi_{323}(\Delta(f)) \circ \Phi^{(2)} &= \Phi^{(2)} \circ \pi_{232}(\Delta(f)) \end{aligned} \tag{2.33}$$

for any $f \in A_q(\mathrm{SL}_4)$. According to (2.14), we set

$$\Phi^{(1)} = R^{(1)} P_{13}, \quad \Phi^{(2)} = R^{(2)} P_{13}, \tag{2.34}$$

where $R^{(1)}$ is (2.19) and $R^{(2)}$ is obtained from it by changing (μ_1, μ_2) to (μ_2, μ_3). The cumbersome upper indices can always be forgotten by specializing the parameters to $\mu_1 = \mu_2 = \mu_3$.

Let $w_0 \in W(\mathrm{SL}_4)$ be the longest element of the Weyl group. We pick two reduced expressions say,

$$w_0 = s_1 s_2 s_3 s_1 s_2 s_1 = s_3 s_2 s_3 s_1 s_2 s_3, \tag{2.35}$$

where the two sides are interchanged by replacing s_i by s_{4-i} and reversing the order. According to Theorem 2.2, we have the equivalence of the two representations of $A_q(\mathrm{SL}_4)$:

$$\pi_{123121} \simeq \pi_{323123}. \tag{2.36}$$

Let P_{ij} and $\Phi^{(1)}_{ijk}, \Phi^{(2)}_{ijk}$ be the transposition P (2.8) and the intertwiners $\Phi^{(1)}, \Phi^{(2)}$ that act on the tensor components specified by the indices. These components must be adjacent (i.e. $j - i = k - j = 1$) to make the relations (2.9) and (2.33) work. With this guideline, one can construct the intertwiners for (2.36) by following the transformation of the reduced expressions by the Coxeter relations

$$s_1 s_3 = s_3 s_1, \quad s_1 s_2 s_1 = s_2 s_1 s_2, \quad s_2 s_3 s_2 = s_3 s_2 s_3.$$

There are two ways to achieve this. In terms of the indices, they look as follows:

$$\begin{array}{llcll}
123\underline{121} & \Phi^{(1)}_{456} & \qquad & 12\underline{31}21 & P_{34} \\
1\underline{232}12 & \Phi^{(2)}_{234} & & \underline{121}321 & \Phi^{(1)}_{123} \\
\underline{13}2\underline{31}2 & P_{12}P_{45} & & 21\underline{232}1 & \Phi^{(2)}_{345} \\
3\underline{121}32 & \Phi^{(1)}_{234} & & 2\underline{13}2\underline{31} & P_{23}P_{56} \\
321\underline{232} & \Phi^{(2)}_{456} & & 23\underline{121}3 & \Phi^{(1)}_{345} \\
32\underline{13}23 & P_{34} & & \underline{232}123 & \Phi^{(2)}_{123} \\
323123 & & & 323123 &
\end{array} \tag{2.37}$$

The underlines indicate the components to which the intertwiners given on the right are to be applied. Since (2.36) is irreducible, we get

$$P_{34}\Phi^{(2)}_{456}\Phi^{(1)}_{234}P_{12}P_{45}\Phi^{(2)}_{234}\Phi^{(1)}_{456} = \Phi^{(2)}_{123}\Phi^{(1)}_{345}P_{23}P_{56}\Phi^{(2)}_{345}\Phi^{(1)}_{123}P_{34}. \tag{2.38}$$

Substituting $\Phi^{(1)}_{ijk} = R^{(1)}_{ijk} P_{jk}$ and $\Phi^{(2)}_{ijk} = R^{(2)}_{ijk} P_{jk}$[2] into this and sending all the P_{ij}'s through to the right, we find that the products of P_{ij}'s correspond to the longest element in the symmetric group $\mathfrak{S}_6$ on the both sides. Thus canceling them out, we obtain

$$R^{(2)}_{356}R^{(1)}_{246}R^{(2)}_{145}R^{(1)}_{123} = R^{(2)}_{123}R^{(1)}_{145}R^{(2)}_{246}R^{(1)}_{356} \tag{2.39}$$

[2] See (2.34). Indices ijk of R here signify the tensor components and should not be confused with those for the matrix elements (2.18).

for the operators acting on $(\mathcal{F}_q)^{\otimes 6}$. When $\mu_1 = \mu_2 = \mu_3$ hence the upper indices can be removed, it reproduces a version of the Zamolodchikov tetrahedron equation [43, 44]. In particular it implies that $\mathscr{R}$ (2.20) satisfies

$$\mathscr{R}_{356}\mathscr{R}_{246}\mathscr{R}_{145}\mathscr{R}_{123} = \mathscr{R}_{123}\mathscr{R}_{145}\mathscr{R}_{246}\mathscr{R}_{356}. \tag{2.40}$$

This is an equality among polynomials of q and free from the other parameters.

Let us write $123121 \to 323123$ to stand for the above calculation leading to the tetrahedron equation (2.39). There are 16 reduced expressions for w_0 in total and one can play the same game with the other 7 pairs. The result is given by

$$\begin{aligned}
121321 \to 321323: &\quad R^{(2)}_{456}R^{(1)}_{236}R^{(2)}_{135}R^{(1)}_{124} = R^{(2)}_{124}R^{(1)}_{135}R^{(2)}_{236}R^{(1)}_{456},\\
123212 \to 232123: &\quad \bar{R}^{(2)}_{321}R^{(2)}_{156}R^{(1)}_{246}R^{(2)}_{345} = R^{(1)}_{345}R^{(2)}_{246}R^{(1)}_{156}\bar{R}^{(1)}_{321},\\
132132 \to 213213: &\quad R^{(2)}_{346}R^{(1)}_{126}\bar{R}^{(1)}_{531}\bar{R}^{(2)}_{542} = \bar{R}^{(1)}_{542}\bar{R}^{(2)}_{531}R^{(2)}_{126}R^{(1)}_{346},\\
132312 \to 231213: &\quad R^{(2)}_{246}R^{(1)}_{136}\bar{R}^{(1)}_{521}\bar{R}^{(2)}_{543} = \bar{R}^{(1)}_{543}\bar{R}^{(2)}_{521}R^{(2)}_{136}R^{(1)}_{246},\\
212321 \to 321232: &\quad R^{(1)}_{234}R^{(2)}_{135}R^{(1)}_{126}\bar{R}^{(1)}_{654} = \bar{R}^{(2)}_{654}R^{(2)}_{126}R^{(1)}_{135}R^{(2)}_{234},\\
213231 \to 312132: &\quad R^{(2)}_{135}R^{(1)}_{146}\bar{R}^{(1)}_{652}\bar{R}^{(2)}_{432} = \bar{R}^{(1)}_{432}\bar{R}^{(2)}_{652}R^{(2)}_{146}R^{(1)}_{135},\\
231231 \to 312312: &\quad R^{(2)}_{134}R^{(1)}_{156}\bar{R}^{(1)}_{642}\bar{R}^{(2)}_{532} = \bar{R}^{(1)}_{532}\bar{R}^{(2)}_{642}R^{(2)}_{156}R^{(1)}_{134},
\end{aligned}$$

where the notation $\bar{R} = R^{-1}$ has been used to uniform the spacing. Thanks to (2.31), we have $R^{(1)}_{ijk} = R^{(1)}_{kji}$ and $R^{(2)}_{ijk} = R^{(2)}_{kji}$. Using this symmetry it can be checked that all the above relations reduce to the single tetrahedron equation (2.39).

One can derive similar compatibility conditions for the intertwiners in $A_q(\mathrm{SL}_n)$ with $n \geq 5$. We expect that they are all attributed to (2.40) in the parameter-free case. For instance for $n = 5$, the longest element is of length 10 and the compatibility is expressed as

$$\mathscr{R}_{123}\mathscr{R}_{145}\mathscr{R}_{246}\mathscr{R}_{356}\mathscr{R}_{178}\mathscr{R}_{279}\mathscr{R}_{389}\mathscr{R}_{470}\mathscr{R}_{580}\mathscr{R}_{690} = \text{product in reverse order},$$

where 0 is the abbreviation of 10. This can be derived by using (2.40) five times.

Setting $q = 0$ in (2.40), we find that the combinatorial 3D R in Remark 2.6 also satisfies the tetrahedron equation

$$\mathcal{R}_{356}\mathcal{R}_{246}\mathcal{R}_{145}\mathcal{R}_{123} = \mathcal{R}_{123}\mathcal{R}_{145}\mathcal{R}_{246}\mathcal{R}_{356}. \tag{2.41}$$

It is an identity of the bijections on subsets of $(\mathbb{Z}_{\geq 0})^6$.

Example 2.8. To demonstrate (2.41), we denote a monomial $|i_1\rangle \otimes \cdots \otimes |i_6\rangle \in (\mathcal{F}_q)^{\otimes 6}$ simply by $|i_1, \ldots, i_6\rangle$. Then the monomial, say, $|314516\rangle$ is transformed as in Figure 1. (In this example $i_1, \ldots, i_6$ remain less than ten, so they are all specified by a single digit.)

$$\begin{tikzcd}
 & {|314516\rangle} \arrow[dl, "\mathcal{R}_{123}"'] \arrow[dr, "\mathcal{R}_{356}"] & \\
{|132516\rangle} \arrow[d, "\mathcal{R}_{145}"'] & & {|311543\rangle} \arrow[d, "\mathcal{R}_{246}"] \\
{|532156\rangle} \arrow[d, "\mathcal{R}_{246}"'] & & {|351147\rangle} \arrow[d, "\mathcal{R}_{145}"] \\
{|512354\rangle} \arrow[dr, "\mathcal{R}_{356}"'] & & {|151327\rangle} \arrow[dl, "\mathcal{R}_{123}"] \\
 & {|515327\rangle} &
\end{tikzcd}$$

FIGURE 1. An example of tetrahedron equation (2.41) for combinatorial 3D R.

The first SW arrow by $\mathcal{R}_{123}$ is due to Example 2.7. If one keeps q generic and lets (2.40) act on the same monomial $|314516\rangle$, each side generates 300 monomials.

2.5. **Classical aspects and triad of 3D *R*.** In terms of Φ (2.14), the combinatorial 3D R is rephrased as the following map on $(\mathbb{Z}_{\geq 0})^3$:

$$\Phi|_{q=0} : (a,b,c) \mapsto (b+c-\min(a,c), \min(a,c), a+b-\min(b,c)).$$

The same map has been introduced in [31, p451] and independence of its compositions corresponding to any reduced expressions of the longest element of $W(\mathrm{SL}_n)$ was utilized.

By regarding a,b,c as indeterminates, this property generalizes to the birational map $(a,b,c) \mapsto \left(\frac{bc}{a+c},\ a+c,\ \frac{ab}{a+c}\right)$. The previous one is reproduced via the *ultradiscretization* (or tropical variable change) $ab \to a+b$ and $a+b \to \min(a,b)$ as pointed out by [42]. See also [8]. Its composition with P_{13} is the map

$$\mathbf{R} : (c,b,a) \mapsto (\tilde{a},\tilde{b},\tilde{c}) = \left(\frac{bc}{a+c},\ a+c,\ \frac{ab}{a+c}\right) \tag{2.42}$$

to be called the *birational* 3D R in the context of the present paper. It is characterized as the unique solution to the matrix equation

$$G_i(a)G_j(b)G_i(c) = G_j(\tilde{a})G_i(\tilde{b})G_j(\tilde{c}) \qquad (|i-j|=1), \tag{2.43}$$

where $G_i(x) = 1 + xE_{i,i+1}$ is a generator of the unipotent subgroup of SL_n. The $\mathbf{R}$ is birational due to $\mathbf{R}^{-1} = \mathbf{R}$. The intertwining relation (2.12) is a quantization of (2.43) (with $(i,j) = (1,2)$). Note that $G_i(a)G_j(b) = G_j(b)G_i(a)$ for $|i-j| > 1$ also holds analogously to the Coxeter relations.

Given a Weyl group element $w \in W(\mathrm{SL}_n)$ (not necessarily longest), assign the matrix $M = G_{i_1}(x_1)\cdots G_{i_r}(x_r)$ to a reduced expression $w = s_{i_1}\cdots s_{i_r}$. Then to any reduced expression $w = s_{j_1}\cdots s_{j_r}$ one can assign the expression $M = G_{j_1}(\tilde{x}_1)\cdots G_{j_r}(\tilde{x}_r)$, where $\tilde{x}_k$ is determined independently of the intermediate steps applying (2.43). This property is the source of the tetrahedron equation for $\mathbf{R}$ and forms a classical (or birational) counterpart of the previous calculation (2.37). In fact, the uniqueness of the map $(a,b,c,d,e,f) \mapsto (\tilde{a},\tilde{b},\tilde{c},\tilde{d},\tilde{e},\tilde{f})$ defined by

$$G_1(a)G_2(b)G_3(c)G_1(d)G_2(e)G_1(f) = G_3(\tilde{a})G_2(\tilde{b})G_3(\tilde{c})G_1(\tilde{d})G_2(\tilde{e})G_3(\tilde{f}) \tag{2.44}$$

proves the *birational tetrahedron equation*

$$\mathbf{R}_{356}\mathbf{R}_{246}\mathbf{R}_{145}\mathbf{R}_{123} = \mathbf{R}_{123}\mathbf{R}_{145}\mathbf{R}_{246}\mathbf{R}_{356}, \tag{2.45}$$

where $\mathbf{R}_{ijk}$ is the one acting on the i,j,k-th components in an array of 6 variables. This is a version of the so called functional tetrahedron equation [19, 35, 20], which is known to allow a more general solution than (2.42) connected to the star-triangle electric circuits transformation.

We have summarized the triad of the 3D R's in Table 1 in Section 1. The tetrahedron equations satisfied by them are given in (2.40), (2.41) and (2.45). The combinatorial one $\mathcal{R}$ shows up either at $q=0$ of the quantum one or ultradiscretization of the birational one. This is a quite analogous feature to 2D. See [27, 13] for example. In the next section, we will add a parallel story corresponding to the third row of the table 1.

3. Sp case

3.1. **Quantized algebra of functions $A_q(\mathrm{Sp}_{2n})$.** We define $A_q(\mathrm{Sp}_{2n})$ following [34], where it was denoted by $\mathrm{Fun}(\mathrm{Sp}_q(n))$. First, we introduce the structure constants $(R_{ij,kl})_{1\le i,j,k,l\le 2n}$ and $C=-C^{-1}=(C_{ij})_{1\le i,j\le 2n}$ by

$$\begin{aligned}
&\sum_{i,j,k,l} R_{ij,kl}E_{ik}\otimes E_{jl} = q\sum_i E_{ii}\otimes E_{ii} + \sum_{i\neq j,j'} E_{ii}\otimes E_{jj} + q^{-1}\sum_i E_{ii}\otimes E_{i'i'}\\
&\qquad\qquad + (q-q^{-1})\sum_{i>j} E_{ij}\otimes E_{ji} - (q-q^{-1})\sum_{i>j}\epsilon_i\epsilon_j q^{\varrho_i-\varrho_j}E_{ij}\otimes E_{i'j'},\\
&C_{ij}=\delta_{i,j'}\epsilon_i q^{\varrho_j},\quad i'=2n+1-i,\quad \epsilon_i=1\,(1\le i\le n),\quad \epsilon_i=-1\,(n<i\le 2n),\\
&(\varrho_1,\ldots,\varrho_{2n})=(n-1,n-2,\ldots,1,0,0,-1,\ldots,-n+1).
\end{aligned}$$

Here the indices are summed over $\{1,2,\ldots,2n\}$ under the specified conditions. The constant $R_{ij,kl}$ is extracted as

$$\sum_{1\le i,j,m,l\le 2n} R_{ij,ml}E_{im}\otimes E_{jl} = q\lim_{x\to\infty} x^{-2}R(x)|_{k=q^{-1}},$$

from the quantum R matrix $R(x)$ [4, 16] for the vector representation of $U_q(C_n^{(1)})$ given in [16, eq.(3.6)]. For example the matrix C for $n=2$ reads

$$C=\begin{pmatrix} 0 & 0 & 0 & q^{-2}\\ 0 & 0 & q^{-1} & 0\\ 0 & -q & 0 & 0\\ -q^2 & 0 & 0 & 0\end{pmatrix}.$$

The quantized algebra of functions $A_q(\mathrm{Sp}_{2n})$ is a Hopf algebra [1] generated by $T=(t_{ij})_{1\le i,j\le 2n}$ with the relations (2.1) and

$$TCT^tC^{-1}=CT^tC^{-1}T=I,\ \text{i.e.}\ \sum_{jkl}C_{jk}C_{lm}t_{ij}t_{lk}=\sum_{jkl}C_{ij}C_{kl}t_{kj}t_{lm}=-\delta_{im}. \tag{3.1}$$

The coproduct is again given by (2.3). We omit the antipode and counit for they will not be used in this paper.

3.2. **Representations of $A_q(\mathrm{Sp}_6)$.** In SL case, we first considered $A_q(\mathrm{SL}_3)$ to determine the intertwiner and then proceeded to $A_q(\mathrm{SL}_4)$ to derive the tetrahedron equation for the purpose of exposition. Here we shorten our presentation by skipping $A_q(\mathrm{Sp}_4)$ and considering $A_q(\mathrm{Sp}_6)$ from the outset since the latter includes the former and presents a generic situation.

Let Osc_q and $\mathcal{F}_q$ be the q-oscillator algebra and the Fock space introduced in (2.4) and (2.5). For distinction we write $\mathrm{Osc}_{q^2}=\langle \mathbf{1},\mathbf{A}^+,\mathbf{A}^-,\mathbf{K}\rangle$, which acts on $\mathcal{F}_{q^2}$. Set

$$q_1=q,\quad q_2=q,\quad q_3=q^2. \tag{3.2}$$

Consider the maps $\pi_i\,(i=1,2,3):\ A_q(\mathrm{Sp}_6)\to\mathrm{Osc}_{q_i}$ which send the generators

$$\begin{pmatrix}
t_{11} & t_{12} & t_{13} & t_{14} & t_{15} & t_{16}\\
t_{21} & t_{22} & t_{23} & t_{24} & t_{25} & t_{26}\\
t_{31} & t_{32} & t_{33} & t_{34} & t_{35} & t_{36}\\
t_{41} & t_{42} & t_{43} & t_{44} & t_{45} & t_{46}\\
t_{51} & t_{52} & t_{53} & t_{54} & t_{55} & t_{56}\\
t_{61} & t_{62} & t_{63} & t_{64} & t_{65} & t_{66}
\end{pmatrix}$$

to the following:

$$\pi_1:\begin{pmatrix} \mu_1\mathbf{a}^- & \alpha_1\mathbf{k} & 0 & 0 & 0 & 0 \\ \beta_1\mathbf{k} & \nu_1\mathbf{a}^+ & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_1\mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_1^{-1}\mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \nu_1^{-1}\mathbf{a}^- & q\beta_1^{-1}\mathbf{k} \\ 0 & 0 & 0 & 0 & q\alpha_1^{-1}\mathbf{k} & \mu_1^{-1}\mathbf{a}^+ \end{pmatrix} \quad (\alpha_1\beta_1=-q\mu_1\nu_1), \tag{3.3}$$

$$\pi_2:\begin{pmatrix} \sigma_2\mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu_2\mathbf{a}^- & \alpha_2\mathbf{k} & 0 & 0 & 0 \\ 0 & \beta_2\mathbf{k} & \nu_2\mathbf{a}^+ & 0 & 0 & 0 \\ 0 & 0 & 0 & \nu_2^{-1}\mathbf{a}^- & q\beta_2^{-1}\mathbf{k} & 0 \\ 0 & 0 & 0 & q\alpha_2^{-1}\mathbf{k} & \mu_2^{-1}\mathbf{a}^+ & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_2^{-1}\mathbf{1} \end{pmatrix} \quad (\alpha_2\beta_2=-q\mu_2\nu_2), \tag{3.4}$$

$$\pi_3:\begin{pmatrix} \rho'\mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \rho\mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_3\mathbf{A}^- & \alpha_3\mathbf{K} & 0 & 0 \\ 0 & 0 & \beta_3\mathbf{K} & \mu_3^{-1}\mathbf{A}^+ & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho^{-1}\mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho'^{-1}\mathbf{1} \end{pmatrix} \quad (\alpha_3\beta_3=-q^2), \tag{3.5}$$

where $\alpha_i, \beta_i, \mu_i\,(i=1,2,3)$, $\sigma_i, \nu_i\,(i=1,2)$, ρ and ρ' are parameters obeying the constraints in the parentheses. One can directly verify

Proposition 3.1. *The maps $\pi_i\,(i=1,2,3)$ are naturally extended to the algebra homomorphisms. The resulting representations of $A_q(\mathrm{Sp}_6)$ on $\mathcal{F}_{q_i}$ are irreducible.*

The representations $A_q(\mathrm{Sp}_6)\to\mathrm{End}(\mathcal{F}_{q_i})$ will also be denoted by $\pi_i\,(i=1,2,3)$.

3.3. **Constraints on parameters.** The Weyl group $W(\mathrm{Sp}_6)=\langle s_1,s_2,s_3\rangle$ is a Coxeter system generated by simple reflections s_1, s_2 and s_3 obeying the relations

$$s_1^2=s_2^2=s_3^2=1,\quad s_1s_3=s_3s_1,\quad s_1s_2s_1=s_2s_1s_2,\quad s_2s_3s_2s_3=s_3s_2s_3s_2. \tag{3.6}$$

Thus, according to Theorem 2.2 (for Sp), one expects the equivalence of the representations

$$\pi_1\otimes\pi_3\simeq\pi_3\otimes\pi_1, \tag{3.7}$$

$$\pi_1\otimes\pi_2\otimes\pi_1\simeq\pi_2\otimes\pi_1\otimes\pi_2, \tag{3.8}$$

$$\pi_2\otimes\pi_3\otimes\pi_2\otimes\pi_3\simeq\pi_3\otimes\pi_2\otimes\pi_3\otimes\pi_2 \tag{3.9}$$

under an appropriate condition on the parameters.

Proposition 3.2. *(i) Eq. (3.7) holds only if*

$$\sigma_1=\pm1,\quad \rho=\rho'. \tag{3.10}$$

(ii) Eq. (3.8) holds only if

$$\sigma_1=\sigma_2,\qquad \alpha_1\beta_1=\alpha_2\beta_2. \tag{3.11}$$

(iii) Under the condition (3.10), the equivalence (3.9) holds only if

$$\rho=\pm1,\quad \alpha_2\beta_2=\pm q, \tag{3.12}$$

where the three signs ±1 in (3.10) and (3.12) can be chosen independently.

Proof. (i) The intertwiner for (3.7) is just the transposition P in (2.8). The relation (2.9) with $(i,j)=(1,3)$ directly leads to (3.10). (ii) and (iii) are derived by investigating the intertwining relations (2.12) for $f\in A_q(\mathrm{Sp}_6)$ and (3.19). $\square$

In view of Proposition 3.2 and (3.3)–(3.5), we set

$$\begin{aligned}
&\pi_1 : \sigma_1 = \sigma, \;\; \alpha_1\beta_1 = -\varepsilon q, \;\; \mu_1\nu_1 = \varepsilon,\\
&\pi_2 : \sigma_2 = \sigma, \;\; \alpha_2\beta_2 = -\varepsilon q, \;\; \mu_2\nu_2 = \varepsilon,\\
&\pi_3 : \rho = \rho', \;\; \alpha_3\beta_3 = -q^2
\end{aligned} \tag{3.13}$$

in the rest of the paper, where the three sign factors

$$\varepsilon = \pm 1, \quad \sigma = \pm 1, \quad \rho = \pm 1 \tag{3.14}$$

can be chosen independently. Given $(\varepsilon, \sigma, \rho) \in \{\pm 1\}^3$, each π_i should be understood as the representation containing two independent parameters α_i and μ_i:

$$\pi_i = \pi_i^{\alpha_i,\mu_i} \; : \; A_q(\mathrm{Sp}_6) \to \mathrm{End}(\mathcal{F}_{q_i}) \quad (i = 1, 2, 3), \tag{3.15}$$

which is defined by (3.3)–(3.5) with (3.13).

3.4. **Intertwiner Φ and R.** Let Φ be the intertwiner for (3.8). It is characterized by formally the same relations as (2.12) and (2.13) with $f \in A_q(\mathrm{Sp}_6)$. As in the SL case (2.14), we introduce $R = \Phi P_{13}$ which satisfies (2.15) for $f \in A_q(\mathrm{Sp}_6)$. It is easy to show

Theorem 3.3. *The $R = (R^{abc}_{ijk})$ is given by*

$$R^{abc}_{ijk} = \varepsilon^j (\sigma\mu_1)^{a-j+k} (\sigma\mu_2)^{b-a-k} \mathcal{R}^{abc}_{ijk}, \tag{3.16}$$

where $\mathcal{R}^{abc}_{ijk}$ is the parameter-free (except q) one specified in (2.20).

Thus the intertwiner R is the same as the SL case up to an overall factor. It satisfies the tetrahedron equation (2.39) if one identifies $R^{(1)}$ with (3.16) and set $R^{(2)} = R^{(1)}|_{(\mu_1,\mu_2)\to(\mu_2,\kappa)}$ for any parameter κ.

Set $\bar{R} = R^{-1} = (\bar{R}^{abc}_{ijk})$. From (2.23) and (3.16), its matrix elements are given by

$$\bar{R}^{abc}_{ijk} = \varepsilon^b (\sigma\mu_1)^{b-a-k} (\sigma\mu_2)^{a-j+k} \mathcal{R}^{abc}_{ijk}. \tag{3.17}$$

3.5. **Intertwiner Ψ and K.** Now we face the new object. Let

$$\Psi : \mathcal{F}_q \otimes \mathcal{F}_{q^2} \otimes \mathcal{F}_q \otimes \mathcal{F}_{q^2} \longrightarrow \mathcal{F}_{q^2} \otimes \mathcal{F}_q \otimes \mathcal{F}_{q^2} \otimes \mathcal{F}_q \tag{3.18}$$

be the intertwiner for (3.9). It is characterized by the following relations:

$$\pi_{3232}(\Delta(f)) \circ \Psi = \Psi \circ \pi_{2323}(\Delta(f)) \quad (\forall f \in A_q(\mathrm{Sp}_6)), \tag{3.19}$$

$$\Psi(|0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle) = |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle, \tag{3.20}$$

where the latter just specifies a normalization. We find it convenient to work with K defined by

$$K = \Psi P_{14} P_{23} : \; \mathcal{F}_{q^2} \otimes \mathcal{F}_q \otimes \mathcal{F}_{q^2} \otimes \mathcal{F}_q \longrightarrow \mathcal{F}_{q^2} \otimes \mathcal{F}_q \otimes \mathcal{F}_{q^2} \otimes \mathcal{F}_q, \tag{3.21}$$

where the composition $P_{14}P_{23} : x \otimes y \otimes z \otimes w \mapsto w \otimes z \otimes y \otimes x$ reverses the order of the 4-fold tensor product. The intertwining relation (3.19) is translated into

$$\pi_{3232}(\Delta(f)) \circ K = K \circ \pi_{3232}(\tilde{\Delta}(f)) \quad (\forall f \in A_q(\mathrm{Sp}_6)), \tag{3.22}$$

where $\tilde{\Delta}(f) = P_{14}P_{23}(\Delta(f))P_{14}P_{23}$, namely,

$$\tilde{\Delta}(t_{ij}) = \sum_{l_1,l_2,l_3} t_{l_3 j} \otimes t_{l_2 l_3} \otimes t_{l_1 l_2} \otimes t_{i l_1}.$$

In Theorem 3.4, we will see that (3.22) becomes independent of the the signs $(\varepsilon, \sigma, \rho)$ and the parameters (α_i, μ_i) if one switches to the "universal part" $\mathcal{K}$ of K by (3.26). The resulting intertwining relations for $\mathcal{K}$ is listed in Appendix A.

Introduce the matrix elements by

$$K(|a\rangle \otimes |i\rangle \otimes |b\rangle \otimes |j\rangle) = \sum_{c,m,d,n} K^{cmdn}_{a\,i\,b\,j} |c\rangle \otimes |m\rangle \otimes |d\rangle \otimes |n\rangle. \tag{3.23}$$

The normalization condition (3.20) becomes $K^{0000}_{0000} = 1$. Let $\mathcal{K} = (\mathcal{K}^{cmdn}_{a\,i\,b\,j})$ be the matrix defined via K as in (3.26). By similar arguments to Proposition 2.4 one can show

$$K^{-1} = K, \qquad \mathcal{K}^{-1} = \mathcal{K}, \tag{3.24}$$

$$(q^4)_c(q^2)_m(q^4)_d(q^2)_n\,\mathcal{K}^{cmdn}_{a\,i\,b\,j} = (q^4)_a(q^2)_i(q^4)_b(q^2)_j\,\mathcal{K}^{a\,i\,b\,j}_{cmdn}. \tag{3.25}$$

Now we present the main formula of the paper.

Theorem 3.4. *The unique solution to the equation (3.22) satisfying $K^{0000}_{0000} = 1$ has the form*

$$K^{cmdn}_{a\,i\,b\,j} = \varepsilon^{m+j}\mu_2^{2d-2b}(\rho\mu_3)^{m-i}\mathcal{K}^{cmdn}_{a\,i\,b\,j}, \tag{3.26}$$

where $\mathcal{K}^{cmdn}_{a\,i\,b\,j}$ is independent of the parameters. It is expressed as

$$\begin{aligned}\mathcal{K}^{cmdn}_{a\,i\,b\,j} &= \delta_{c+m+d,a+i+b}\delta_{d+n-c,b+j-a}\frac{(q^4)_a}{(q^4)_c}\sum_{\alpha,\beta,\gamma}\frac{(-1)^{\alpha+\gamma}}{(q^4)_{d-\beta}}q^{\phi_1}\\ &\quad\times\mathcal{K}^{a,i+b-\alpha-\beta-\gamma,0,j+b-\alpha-\beta-\gamma}_{c,m+d-\alpha-\beta-\gamma,0,n+d-\alpha-\beta-\gamma}\begin{Bmatrix} b, d-\beta, i+b-\alpha-\beta, j+b-\alpha-\beta \\ \alpha,\beta,\gamma, m-\alpha, n-\alpha, b-\alpha-\beta, d-\beta-\gamma\end{Bmatrix},\\ \phi_1 &= \alpha(\alpha+2d-2\beta-1)+(2\beta-d)(m+n+d)+\gamma(\gamma-1)-b(i+j+b),\end{aligned} \tag{3.27}$$

where the sum is over $\alpha,\beta,\gamma\in\mathbb{Z}_{\geq 0}$, which is actually finite. The $\mathcal{K}$ in the sum is given by

$$\begin{aligned}\mathcal{K}^{cm0n}_{a\,i\,0\,j} &= \delta_{c+m,a+i}\delta_{n-c,j-a}\sum_{\lambda}(-1)^{m+\lambda}\frac{(q^4)_{c+\lambda}}{(q^4)_c}q^{\phi_2}\begin{Bmatrix} i, j \\ \lambda, j-\lambda, m-\lambda, i-m+\lambda\end{Bmatrix},\\ \phi_2 &= (a+c+1)(m+j-2\lambda)+m-j,\end{aligned} \tag{3.28}$$

where the sum is over $\lambda\in\mathbb{Z}_{\geq 0}$, which is actually finite.

A proof of Theorem 3.4 is available in Appendix B. At $d = b = 0$, the formula (3.27) reduces to (3.25). Note that

$$\mathcal{K}^{cmdn}_{a\,i\,b\,j} = 0 \ \text{ unless } c+m+d = a+i+b \text{ and } d+n-c = b+j-a. \tag{3.29}$$

As with $\mathcal{R}$, this property will also be referred as the conservation law. We have separated the result into (3.27) and (3.28) as the full formula obtained by their composition is rather bulky.

In (3.28), the quantity $\left\{\begin{smallmatrix}\cdots\\ \cdots\end{smallmatrix}\right\}$ is a product of two q^2-binomial coefficients, therefore $\mathcal{K}^{cm0n}_{a\,i\,0\,j}$ is a Laurent polynomial of q. On the other hand, (3.27) only tells that $\mathcal{K}^{cmdn}_{a\,i\,b\,j}$ is a rational function of q in general. Our second main result concerns this point and exhibits a remarkable feature analogous to $\mathcal{R}$ mentioned in Remark 2.6.

Theorem 3.5. *(i) The matrix elements $\mathcal{K}^{cmdn}_{a\,i\,b\,j}$ in Theorem 3.4 are polynomials in q with integer coefficients.*

(ii) Set

$$\mathcal{K} = (\mathcal{K}^{cmdn}_{a\,i\,b\,j}) := \mathcal{K}|_{q=0}, \tag{3.30}$$

hence $\mathcal{K}^{cmdn}_{a\,i\,b\,j} = \mathcal{K}^{cmdn}_{a\,i\,b\,j}|_{q=0}$. Then it is given explicitly as

$$\begin{aligned}&\mathcal{K}^{cmdn}_{a\,i\,b\,j} = \delta_{c+m+d,a+i+b}\delta_{d+n-c,b+j-a}\,\delta_{a,c'}\delta_{i,m'}\delta_{b,d'}\delta_{j,n'},\\ &c' = x+c+m-n, \ \ m' = d-x+n-\min(c,d+x),\\ &d' = \min(c,d+x), \ \ n' = m+(d+x-c)_+, \ \ x = (d-c+(n-m)_+)_+,\end{aligned} \tag{3.31}$$

where the symbol $(y)_+$ is defined in Remark 2.6.

A proof of Theorem 3.5 is outlined in Appendix C. We note that ϕ_1 and ϕ_2 in (3.27) and (3.28) can become negative in general, so the claim (i) implies nontrivial cancellations. It is an interesting problem to construct an explicit formula of $\mathcal{K}^{cmdn}_{a\,i\,b\,j}$ in which its polynomiality is manifest. From (3.27) and (3.28) the claim (i) can be refined to $\mathcal{K}^{cmdn}_{a\,i\,b\,j} \in q^\eta\mathbb{Z}[q^2]$, where $\eta = 0, 1$ is specified by $\eta \equiv ij+mn \mod 2$.

From the claim (i) and (3.24) it follows that $\mathcal{K} = \mathcal{K}^{-1}$. Thus replacing K with $\mathcal{K}$ in (3.23) defines a bijection. Put in another word, $\mathcal{K} : (c, m, d, n) \mapsto (c', m', d', n')$ is a bijection on each finite set specified by the values of conserved quantities $\{(c, m, d, n) \in (\mathbb{Z}_{\geq 0})^4 \,|c + m + d = \mathrm{const}, d + n - c = \mathrm{const}\}$. In fact the property $c', m', d', n' \in \mathbb{Z}_{\geq 0}$ and the conservation law $c+m+d = c'+m'+d'$, $d+n-c = d'+n'-c'$ can easily be confirmed. We call $\mathcal{K}$ *combinatorial* 3D K. See [28] for an analogous object in 2D and its application. We shall see another origin of the piecewise linear formula (3.31) in Section 3.7.

Example 3.6. The following is the list of all the nonzero $\mathcal{K}^{cmdn}_{2\,1\,1\,0}$.

$$
\begin{aligned}
\mathcal{K}^{1300}_{2110} &= q^8(1 - q^8),\\
\mathcal{K}^{2110}_{2110} &= -q^4(1 - q^8 + q^{14}),\\
\mathcal{K}^{2201}_{2110} &= -q^6(1 + q^2)(1 - q^2 + q^4 - q^6 - q^{10}),\\
\mathcal{K}^{3011}_{2110} &= 1 - q^8 + q^{14},\\
\mathcal{K}^{3102}_{2110} &= -q^{10}(1 - q + q^2)(1 + q + q^2),\\
\mathcal{K}^{4003}_{2110} &= q^4.
\end{aligned}
$$

Thus $\mathcal{K}^{cmdn}_{2\,1\,1\,0} = \delta_{c,3}\delta_{m,0}\delta_{d,1}\delta_{n,1}$ in agreement with (3.31).

3.6. **Relation involving R and K.** Let $w_0 \in W(\mathrm{Sp}_6)$ be the longest element of the Weyl group and consider the two reduced expressions

$$
w_0 = s_1s_2s_3s_2s_1s_2s_3s_2s_3 = s_3s_2s_3s_2s_1s_2s_3s_2s_1, \tag{3.32}
$$

where the order of the simple reflections are opposite in the two sides. According to Theorem 2.2 for Sp case, we have the equivalence of the two representations of $A_q(\mathrm{Sp}_6)$:

$$
\pi_{123212323} \simeq \pi_{323212321}. \tag{3.33}
$$

Let P_{ij}, Φ_{ijk} and Ψ_{ijkl} be the transposition P (2.8), the intertwiner Φ in Section 3.4 and the intertwiner Ψ (3.18) that act on the tensor components specified by the indices. As in Section 2.4, one can construct two intertwiners for (3.33) by consulting the Coxeter relations (3.6).

$12321\underline{2323}$	Ψ_{6789}	$123\underline{212}323$	Φ^{-1}_{456}
$1232\underline{13}232$	P_{56}	$12\underline{31}2\underline{13}23$	$P_{34}P_{67}$
$1\underline{2323}1232$	Ψ_{2345}	$\underline{121}323123$	Φ_{123}
$1323\underline{212}32$	Φ^{-1}_{567}	$21\underline{2323}123$	Ψ_{3456}
$\underline{13}2\underline{31}2\underline{13}2$	$P_{12}P_{45}P_{78}$	$21323\underline{212}3$	Φ^{-1}_{678}
$3\underline{121}32312$	Φ_{234}	$2\underline{13}2\underline{31}2\underline{13}$	$P_{23}P_{56}P_{89}$
$321\underline{2323}12$	Ψ_{4567}	$23\underline{121}3231$	Φ_{345}
$321323\underline{212}$	Φ^{-1}_{789}	$2321\underline{2323}1$	Ψ_{5678}
$32\underline{13}2\underline{31}21$	$P_{34}P_{67}$	$232\underline{13}2321$	P_{45}
$323\underline{121}321$	Φ_{456}	$\underline{2323}12321$	Ψ_{1234}
323212321		323212321	

The underlines are assigned in the same manner as in (2.37). Thus we get

$$
\begin{aligned}
&\Phi_{456}P_{34}P_{67}\Phi^{-1}_{789}\Psi_{4567}\Phi_{234}P_{12}P_{45}P_{78}\Phi^{-1}_{567}\Psi_{2345}P_{56}\Psi_{6789}\\
=&\Psi_{1234}P_{45}\Psi_{5678}\Phi_{345}P_{23}P_{56}P_{89}\Phi^{-1}_{678}\Psi_{3456}\Phi_{123}P_{34}P_{67}\Phi^{-1}_{456}.
\end{aligned}
\tag{3.34}
$$

Substituting ($\bar{R}$ is a shorthand for R^{-1} as in Section 2.4)

$$
\Phi_{ijk} = R_{ijk}P_{jk}, \quad \Phi^{-1}_{ijk} = P_{jk}\bar{R}_{ijk}, \quad \Psi_{ijkl} = K_{ijkl}P_{il}P_{jk}
$$

into (3.34) and sending all the P_{ij}'s through to the right, we find that the products of P_{ij}'s correspond to the longest element in the symmetric group $\mathfrak{S}_9$ on the both sides. Canceling them out, we obtain

$$R_{456}\bar{R}_{984}K_{3579}R_{269}\bar{R}_{852}K_{1678}K_{1234} = K_{1234}K_{1678}R_{258}\bar{R}_{962}K_{3579}R_{489}\bar{R}_{654} \tag{3.35}$$

for the operators acting on $\pi_{323212321}$. Namely, (3.35) is an equality in $\mathrm{End}(\mathcal{F}_{q^2} \otimes \mathcal{F}_q \otimes \mathcal{F}_{q^2} \otimes \mathcal{F}_q \otimes \mathcal{F}_q \otimes \mathcal{F}_q \otimes \mathcal{F}_{q^2} \otimes \mathcal{F}_q \otimes \mathcal{F}_q)$. We call (3.35) (and (3.36) given below as well) the 3D reflection equation.

There are 42 reduced expressions for the longest element w_0. We can ask if one obtains other relations than (3.35) from different reduced expressions. The answer is negative as in the tetrahedron equation (2.39). Namely, all the other relations reduce to (3.35), since all 42 reduced expressions of w_0 essentially appear in either course from 123212323 to 323212321 to get (3.34).

Let us discuss the physical interpretation of the construction here. As mentioned in the introduction, the relevant system is the factorized scattering of strings in 3D [43, 44] under the presence of boundary reflections [14]. Our 3D reflection equation (3.35) is a constant version of the tetrahedron reflection equation [14]. This can be seen by relabeling the space indices in (3.35) as $(1, 2, 3, 4, 5, 6, 7, 8, 9) \to (\overline{x}, 6, \overline{y}, 5, 4, 3, \overline{z}, 2, 1)$. The resulting equation essentially coincides with [14, eq.(17)]. The spaces $2, 4, 5, 6, 8, 9$ in (3.35) correspond to $\mathcal{F}_q$ and are attached with strings. The other spaces labeled by $1, 3$ and 7 are $\mathcal{F}_{q^2}$, hence touched upon only by K's. They represent a physical degree of freedom living on the boundary which are subject to quantum transitions when reflecting back strings.

The fundamental representations π_1, π_2, π_3 of $A_q(\mathrm{Sp}_6)$ in this paper correspond to the generators $A_1, \ldots, A_6, B^{\overline{u}}$ in the 3D analogue of the Zamolodchikov algebra [14] as

$$\pi_1 \to A_\bullet, \;\; \pi_2 \to A_\bullet, \;\; \pi_3 \to B^\bullet.$$

In fact, consider for instance the LHS of (3.33). We let the ninth order tensor product correspond to a word of A_i's and $B^{\overline{u}}$ by the above rule as

$$\pi_1 \otimes \pi_2 \otimes \pi_3 \otimes \pi_2 \otimes \pi_1 \otimes \pi_2 \otimes \pi_3 \otimes \pi_2 \otimes \pi_3 \to A_1A_2B^{\overline{z}}A_3A_4A_5B^{\overline{y}}A_6B^{\overline{x}},$$

where indices of A (B) are assigned consecutively (inverse alphabetically). The RHS reproduces the element in [14, (c) p434]. Similarly the intertwining relations of R (2.15) and K (3.22) correspond to the 3D Zamolodchikov algebra [14, eq.(4)] and the boundary reflection algebra [14, eq.(9)], respectively. The reordering process of $A_1A_2B^{\overline{z}}A_3A_4A_5B^{\overline{y}}A_6B^{\overline{x}}$ is translated into the composition of intertwiners K and R as demonstrated after (3.33). Physically A_i represents a straight string moving in 3D or the world sheet generated by it, and $B^{\overline{u}}$ denotes a reflection by the boundary. The three A_i's among the six correspond to the strings heading toward the boundary and the other three to those going off the boundary after the reflections represented by the three $B^{\overline{u}}$'s.

Now we turn to a combinatorial aspect of the 3D reflection equation (3.35) . When all the parameters are removed, it becomes

$$\mathcal{R}_{456}\mathcal{R}_{489}\mathcal{K}_{3579}\mathcal{R}_{269}\mathcal{R}_{258}\mathcal{K}_{1678}\mathcal{K}_{1234} = \mathcal{K}_{1234}\mathcal{K}_{1678}\mathcal{R}_{258}\mathcal{R}_{269}\mathcal{K}_{3579}\mathcal{R}_{489}\mathcal{R}_{456}, \tag{3.36}$$

where we have applied $\mathcal{R}^{-1} = \mathcal{R}$ (2.23) and $\mathcal{R}_{ijk} = \mathcal{R}_{kji}$ due to (2.31). This is an identity between polynomials of q. Setting $q = 0$ further and invoking Theorem 3.5 and Remark 2.6, we find that the combinatorial 3D R and the combinatorial 3D K still satisfy

$$\mathcal{R}_{456}\mathcal{R}_{489}\mathcal{K}_{3579}\mathcal{R}_{269}\mathcal{R}_{258}\mathcal{K}_{1678}\mathcal{K}_{1234} = \mathcal{K}_{1234}\mathcal{K}_{1678}\mathcal{R}_{258}\mathcal{R}_{269}\mathcal{K}_{3579}\mathcal{R}_{489}\mathcal{R}_{456}. \tag{3.37}$$

This is an identity of bijections between finite subsets of $(\mathbb{Z}_{\geq 0})^9$, which may be called the combinatorial 3D reflection equation. It is an interesting problem whether it leads to a 3D generalization of the result like [28].

Example 3.7. To demonstrate (3.37), we employ the same convention as in Example 2.8. Then the monomial, say, $|211034212\rangle$ is transformed as in Figure 2. The first SW arrow by $\mathcal{K}_{1234}$ is due to Example 3.6.

$$
\begin{array}{ccc}
 & |211034212\rangle & \\
\mathcal{K}_{1234} \swarrow & & \searrow \mathcal{R}_{456} \\
|301134212\rangle & & |211307212\rangle \\
\mathcal{K}_{1678} \downarrow & & \downarrow \mathcal{R}_{489} \\
|601131242\rangle & & |211207221\rangle \\
\mathcal{R}_{258} \downarrow & & \downarrow \mathcal{K}_{3579} \\
|631101272\rangle & & |212207123\rangle \\
\mathcal{R}_{269} \downarrow & & \downarrow \mathcal{R}_{269} \\
|621102271\rangle & & |272201129\rangle \\
\mathcal{K}_{3579} \downarrow & & \downarrow \mathcal{R}_{258} \\
|622102173\rangle & & |252221109\rangle \\
\mathcal{R}_{489} \downarrow & & \downarrow \mathcal{K}_{1678} \\
|622702119\rangle & & |352220119\rangle \\
\mathcal{R}_{456} \searrow & & \swarrow \mathcal{K}_{1234} \\
 & |622520119\rangle &
\end{array}
$$

FIGURE 2. An example of the 3D reflection equation (3.37) for the combinatorial 3D R and K.

3.7. **Birational 3D *K*.** Introduce the upper triangular matrices

$$
X_1(z) = \begin{pmatrix} 1 & z & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & -z \\ & & & 1 \end{pmatrix}, \qquad X_2(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ & 1 & 2z & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix},
$$

$$
Y_1(z) = \begin{pmatrix} 1 & z & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 \\ & & 1 & 0 & 0 \\ & & & 1 & -z \\ & & & & 1 \end{pmatrix}, \quad Y_2(z) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ & 1 & z & -z^2/2 & 0 \\ & & 1 & -z & 0 \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix},
$$

where blanks signify 0 and z is a parameter. The matrix $X_i(z)$ is a generator of the unipotent subgroup of Sp_4. Similarly $Y_i(z)$ is the one for SO_5. They are associated with two realizations of the Lie group corresponding to $\mathrm{Sp}_4 \simeq \mathrm{SO}_5$. The matrices $X_2(z)$ and $Y_1(z)$ correspond to the long simple root, so the role of indices 1 and 2 are interchanged in the two pictures. They satisfy $X_i(z)^{-1} = X_i(-z)$ and $Y_i(z)^{-1} = Y_i(-z)$. By a direct calculation one can establish

Theorem 3.8. *Given indeterminates (a, b, c, d), each of the two matrix equations*

$$
X_2(a)X_1(b)X_2(c)X_1(d) = X_1(\tilde{a})X_2(\tilde{b})X_1(\tilde{c})X_2(\tilde{d}), \tag{3.38}
$$

$$
Y_1(a)Y_2(b)Y_1(c)Y_2(d) = Y_2(\tilde{a})Y_1(\tilde{b})Y_2(\tilde{c})Y_1(\tilde{d}) \tag{3.39}
$$

for $(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})$ has the unique solution

$$
\begin{gathered}
\tilde{a} = \frac{bcd}{A}, \quad \tilde{b} = \frac{A^2}{B}, \quad \tilde{c} = \frac{B}{A}, \quad \tilde{d} = \frac{ab^2c}{B}, \\
A = ab + ad + cd, \quad B = ab^2 + 2abd + ad^2 + cd^2.
\end{gathered} \tag{3.40}
$$

Define a map

$$\mathbf{K} : (d, c, b, a) \mapsto (\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}) \tag{3.41}$$

in terms of (3.40), where the reason for not (a, b, c, d) but (d, c, b, a) is to fit $K = \Psi P_{14} P_{23}$ in (3.21). It is easy to see $\mathbf{K}^{-1} = \mathbf{K}$, hence $\mathbf{K}$ is birational. We call $\mathbf{K}$ the *birational* 3D K. The intertwining relation (3.19) is a quantization of (3.38). (For Sp_4, π_{3232} therein should read π_{2121}.) It satisfies the formally identical equation with (3.36):

$$\mathbf{R}_{456}\mathbf{R}_{489}\mathbf{K}_{3579}\mathbf{R}_{269}\mathbf{R}_{258}\mathbf{K}_{1678}\mathbf{K}_{1234} = \mathbf{K}_{1234}\mathbf{K}_{1678}\mathbf{R}_{258}\mathbf{R}_{269}\mathbf{K}_{3579}\mathbf{R}_{489}\mathbf{R}_{456}. \tag{3.42}$$

This is an equality of the birational maps on 9 variables which can be directly checked. Alternatively it can also be derived following the argument similar to (2.44)–(2.45).

The birational 3D K tends to the combinatorial 3D K via the ultradiscretization. In fact by the tropical variable change $\alpha\beta \to \alpha + \beta$ and $\alpha + \beta \to \min(\alpha, \beta)$, the formulas (3.40)–(3.41) exactly reproduce the piecewise-linear map $\mathcal{K} : (c, m, d, n) \mapsto (c', m', d', n')$ in (3.31). Thus we have realized the triad of 3D K's in Table 1 all satisfying the 3D reflection equations (3.36), (3.37) and (3.42).

4. Type B and F_4 cases

From the construction in the preceding sections for algebras of type A and C, it is quite possible to infer the situation in type B and F_4. In this section we discuss them without a proof or concrete realizations of the representations. We shall only be concerned with the parameter-free part of the intertwiners like $\mathcal{R}$ and $\mathcal{K}$ which are polynomials of q only. The minimal and generic situation for the B series takes place for B_3. We list the relevant Dynkin diagrams in Figure 3.

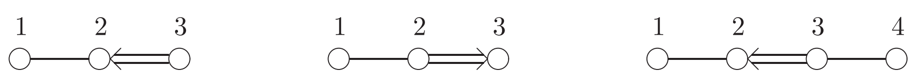

Figure 3. Dynkin diagrams of C_3 (left), B_3 (center) and F_4 (right). Enumeration of vertices for F_4 agrees with [9] which is opposite to [17].

Type B case. The Weyl group $W(B_3)$ is isomorphic to $W(C_3)$. Thus we should have the equivalence (3.7)–(3.9) for the irreducible representations $\pi_i = \pi_i^B (i = 1, 2, 3)$ of the quantized algebra of functions $A_q(\mathrm{SO}_7)$, where SO_7 is the Lie group corresponding to B_3. Note that locally in the Dynkin diagrams, the role of the indices 2 and 3 are interchanged between B_3 and C_3. From this fact and (3.22), the intertwiner $\mathcal{K}^B$ satisfying $(\pi^B_{3232}\Delta) \circ \mathcal{K}^B = \mathcal{K}^B \circ (\pi^B_{3232}\tilde{\Delta})$ should be obtained from the type C case $\mathcal{K}$ as

$$\mathcal{K}^B_{1234} = P_{14}P_{23}\mathcal{K}_{1234}P_{23}P_{14} = \mathcal{K}_{4321} \in \mathrm{End}(\mathcal{F}_q \otimes \mathcal{F}_{q^2} \otimes \mathcal{F}_q \otimes \mathcal{F}_{q^2}). \tag{4.1}$$

Here the second equality just means that its RHS is the standard notation for the middle object acting on the tensor components labeled with $1, 2, 3$ and 4.

Now we proceed to the intertwiner $\mathcal{R}^B$ satisfying $(\pi^B_{212}\Delta) \circ \mathcal{R}^B = \mathcal{R}^B \circ (\pi^B_{121}\tilde{\Delta})$. Since the segment of the Dynkin diagrams between the vertices 1 and 2 are the same for B_3 and C_3, we expect that $\mathcal{R}^B$ is obtained from the type C case (hence A case) $\mathcal{R}$ as

$$\mathcal{R}^B = \mathcal{S} := \mathcal{R}|_{q \to q^2} \in \mathrm{End}(\mathcal{F}_{q^2} \otimes \mathcal{F}_{q^2} \otimes \mathcal{F}_{q^2}). \tag{4.2}$$

Here the replacement $q \to q^2$ reflects the squared length of the simple roots attached to the vertices 1 and 2 compared with 3. It is an opposite of (3.2).

To summarize so far, we conjecture that the intertwiners $\mathcal{K}^B$ and $\mathcal{R}^B$ for $A_q(\mathrm{SO}_7)$ are obtained from the corresponding objects in $A_q(\mathrm{Sp}_6)$ by the simple prescriptions (4.1) and (4.2).

Now we consider the 3D reflection equations. Due to $W(B_3) \simeq W(C_3)$, the intertwiners $\mathcal{K}^B$ and $\mathcal{R}^B$ should fulfill exactly the same relation as (3.36). In other words, (3.36) should survive under the replacement $(\mathcal{R}, \mathcal{K}) \to (\mathcal{R}^B, \mathcal{K}^B)$. Thus we conjecture that

$$\mathcal{S}_{456}\mathcal{S}_{489}\mathcal{K}_{9753}\mathcal{S}_{269}\mathcal{S}_{258}\mathcal{K}_{8761}\mathcal{K}_{4321} = \mathcal{K}_{4321}\mathcal{K}_{8761}\mathcal{S}_{258}\mathcal{S}_{269}\mathcal{K}_{9753}\mathcal{S}_{489}\mathcal{S}_{456} \tag{4.3}$$

holds in $\mathrm{End}(\mathcal{F}_q \otimes \mathcal{F}_{q^2} \otimes \mathcal{F}_q \otimes \mathcal{F}_{q^2} \otimes \mathcal{F}_{q^2} \otimes \mathcal{F}_{q^2} \otimes \mathcal{F}_q \otimes \mathcal{F}_{q^2} \otimes \mathcal{F}_{q^2})$. This is a yet independent relation from (3.36) to be called the *3D reflection equation of type B*. The previous one (3.36) is of type C in this context. We have checked (4.3) by computer for several examples. For instance when the both sides act on the monomial $|112111111\rangle$ specified by the occupation numbers of q-oscillators, they generate the same vector consisting of 1410 monomials.

F_4 case. We let $A_q(\mathrm{F}_4)$ denote the quantized algebra of functions on the Lie group corresponding to F_4. Let π_i be its irreducible representation attached to the vertex i of the Dynkin diagram in Figure 3. We expect that it is realized in terms of the q-oscillators as

$$\pi_i : A_q(\mathrm{F}_4) \to \mathrm{End}(\mathcal{F}_{q_i}) \quad \text{with} \;\; (q_1, q_2, q_3, q_4) = (q, q, q^2, q^2).$$

The q_i here is a natural prolongation of (3.2) reflecting the squared length of the simple roots. The Coxeter relations for the simple reflections $s_i \in W(\mathrm{F}_4)$ are given by

$$s_1s_2s_1 = s_2s_1s_2, \quad s_2s_3s_2s_3 = s_3s_2s_3s_2, \quad s_3s_4s_3 = s_4s_3s_4 \tag{4.4}$$

in addition to the 'trivial' ones $s_i^2 = 1$ and $s_is_j = s_js_i$ for $|i-j| > 1$. Thus one should have the equivalence between the corresponding tensor products of π_i's [39, 40]. Introduce the three kinds of intertwiners $\mathcal{R}^F, \mathcal{K}^F$ and $\mathcal{S}^F$ characterized up to normalization by

$$\begin{aligned}
(\pi_{212}\Delta) \circ \mathcal{R}^F &= \mathcal{R}^F \circ (\pi_{121}\tilde{\Delta}),\\
(\pi_{3232}\Delta) \circ \mathcal{K}^F &= \mathcal{K}^F \circ (\pi_{3232}\tilde{\Delta}),\\
(\pi_{434}\Delta) \circ \mathcal{S}^F &= \mathcal{S}^F \circ (\pi_{343}\tilde{\Delta}).
\end{aligned} \tag{4.5}$$

From the Dynkin diagrams in Figure 3 and the discussion for type B, it is natural to conjecture that they are simply related to the preceding ones as

$$\mathcal{R}^F = \mathcal{R}, \quad \mathcal{K}^F = \mathcal{K}, \quad \mathcal{S}^F = \mathcal{S}. \tag{4.6}$$

The RHSs have been encountered first for type A, C and B and described explicitly in (2.20), Theorem 3.4 and (4.2), respectively.

What about the identities analogous to the 3D reflection equations? We can follow the same argument as before to derive them from a reduced word for the longest element $w_0 \in W(\mathrm{F}_4)$ by reversing it in two ways. The length of w_0 is 24 and there are 2144892 reduced expressions for it. We have picked up

$$s_4s_3s_4s_2s_3s_4s_2s_3s_2s_1s_2s_3s_4s_2s_3s_1s_2s_3s_4s_1s_2s_3s_2s_1 \tag{4.7}$$

and reversed the ordering via the Coxeter relations (4.4). By representing the procedure in terms of the intertwiners (4.6) we find that the consistency is expressed as

$$\begin{aligned}
&\mathcal{S}_{14,15,16}\mathcal{S}_{9,11,16}\mathcal{K}_{16,10,8,7}\mathcal{K}_{9,13,15,17}\mathcal{S}_{4,5,16}\mathcal{R}_{7,12,17}\mathcal{S}_{1,2,16}\mathcal{R}_{6,10,17}\mathcal{S}_{9,14,18}\mathcal{K}_{1,3,5,17}\\
&\times \mathcal{S}_{11,15,18}\mathcal{K}_{18,12,8,6}\mathcal{S}_{1,4,18}\mathcal{S}_{1,8,15}\mathcal{R}_{7,13,19}\mathcal{K}_{1,6,11,19}\mathcal{K}_{4,12,15,19}\mathcal{R}_{3,10,19}\mathcal{S}_{4,8,11}\mathcal{K}_{1,7,14,20}\\
&\times \mathcal{S}_{2,5,18}\mathcal{R}_{6,13,20}\mathcal{R}_{3,12,20}\mathcal{S}_{1,9,21}\mathcal{K}_{2,10,15,20}\mathcal{S}_{4,14,21}\mathcal{K}_{21,13,8,3}\mathcal{S}_{2,11,21}\mathcal{S}_{2,8,14}\mathcal{R}_{6,7,22}\\
&\times \mathcal{K}_{2,3,4,22}\mathcal{S}_{5,15,21}\mathcal{K}_{11,13,14,22}\mathcal{R}_{10,12,22}\mathcal{K}_{2,6,9,23}\mathcal{R}_{3,7,23}\mathcal{R}_{19,20,22}\mathcal{K}_{16,17,18,22}\mathcal{R}_{10,13,23}\mathcal{K}_{5,12,14,23}\\
&\times \mathcal{R}_{3,6,24}\mathcal{K}_{16,19,21,23}\mathcal{K}_{4,7,9,24}\mathcal{R}_{17,20,23}\mathcal{K}_{5,10,11,24}\mathcal{R}_{12,13,24}\mathcal{R}_{17,19,24}\mathcal{K}_{18,20,21,24}\mathcal{S}_{5,8,9}\mathcal{R}_{22,23,24}\\
&= \text{product in reverse order.}
\end{aligned} \tag{4.8}$$

Here we have already applied the properties $\mathcal{R}^{-1} = \mathcal{R}$ (2.23) and $\mathcal{R}_{i,j,k} = \mathcal{R}_{k,j,i}$ (2.31). Each side consists of 16 $\mathcal{R}$'s, 16 $\mathcal{S}$'s and 18 $\mathcal{K}$'s (3 of them having decreasing order of indices) amounting to 50 factors in total. So things get monstrous somewhat as is usual for exceptional Lie algebras, and a physical interpretation seems formidable for (4.8). However it should be emphasized that

its validity is a *corollary* of [39, 40] provided that $\mathcal{R}, \mathcal{K}$ and $\mathcal{S}$ are really the intertwiners $\mathcal{R}^F, \mathcal{K}^F$ and $\mathcal{S}^F$ for $A_q(\mathrm{F}_4)$ characterized by (4.5). This last point, i.e. (4.6) is the only conjectural aspect in (4.8). Again we have confirmed it by computer in several examples, which are limited however considerably to small ones. For instance when the both sides act on the monomial $|111101101010101102110101\rangle$, they both generate the same vector consisting of 533 monomials at least mod $q^6\mathbb{Z}[q]$.

Besides the conjecture, we close with two questions which are yet to be answered. First, can any other consistency relation involving $\mathcal{R}, \mathcal{K}$ and $\mathcal{S}$, say those stemming from other reduced expressions, be attributed to (4.8)? For $A_q(\mathrm{Sp}_6)$ the analogous question had a positive answer. See the remark after (3.35). Second, can (4.8) be attributed to a composition of the tetrahedron equations of $\mathcal{R}, \mathcal{S}$ and the 3D reflection equations of type C (3.36) and type B (4.3)? We hope to report on these issues together with $D_n, E_{6,7,8}$ and G_2 cases in a separate publication.

Appendix A. Intertwining relations for $\mathcal{K}$

Let $\langle rs\rangle$ be the intertwining relation for $\mathcal{K}$ obtained by substituting (3.26) into (3.22) with the choice $f = t_{rs}$. They are independent of the parameters other than q. The relation $\langle rs\rangle$ holds trivially as $0=0$ unless $2 \le r, s \le 5$. The nontrivial cases are given as follows.

$$
\begin{aligned}
\langle 22\rangle:\quad & [\mathbf{1}\otimes\mathbf{a}^-\otimes\mathbf{1}\otimes\mathbf{a}^- - q\mathbf{1}\otimes\mathbf{k}\otimes\mathbf{A}^-\otimes\mathbf{k},\, \mathcal{K}] = 0,\\
\langle 23\rangle:\quad & (\mathbf{1}\otimes\mathbf{a}^-\otimes\mathbf{1}\otimes\mathbf{k} + \mathbf{1}\otimes\mathbf{k}\otimes\mathbf{A}^-\otimes\mathbf{a}^+)\mathcal{K}\\
& = \mathcal{K}(\mathbf{A}^-\otimes\mathbf{a}^+\otimes\mathbf{A}^-\otimes\mathbf{k} + \mathbf{A}^-\otimes\mathbf{k}\otimes\mathbf{1}\otimes\mathbf{a}^- - q^2\mathbf{K}\otimes\mathbf{a}^-\otimes\mathbf{K}\otimes\mathbf{k}),\\
\langle 24\rangle:\quad & (\mathbf{1}\otimes\mathbf{k}\otimes\mathbf{K}\otimes\mathbf{a}^-)\mathcal{K} = \mathcal{K}(\mathbf{A}^+\otimes\mathbf{a}^-\otimes\mathbf{K}\otimes\mathbf{k} + \mathbf{K}\otimes\mathbf{a}^+\otimes\mathbf{A}^-\otimes\mathbf{k} + \mathbf{K}\otimes\mathbf{k}\otimes\mathbf{1}\otimes\mathbf{a}^-),\\
\langle 25\rangle:\quad & [\mathbf{1}\otimes\mathbf{k}\otimes\mathbf{K}\otimes\mathbf{k}, \mathcal{K}] = 0,\\
\langle 32\rangle:\quad & (\mathbf{A}^-\otimes\mathbf{a}^+\otimes\mathbf{A}^-\otimes\mathbf{k} + \mathbf{A}^-\otimes\mathbf{k}\otimes\mathbf{1}\otimes\mathbf{a}^- - q^2\mathbf{K}\otimes\mathbf{a}^-\otimes\mathbf{K}\otimes\mathbf{k})\mathcal{K}\\
& = \mathcal{K}(\mathbf{1}\otimes\mathbf{a}^-\otimes\mathbf{1}\otimes\mathbf{k} + \mathbf{1}\otimes\mathbf{k}\otimes\mathbf{A}^-\otimes\mathbf{a}^+),\\
\langle 33\rangle:\quad & [\mathbf{A}^-\otimes\mathbf{a}^+\otimes\mathbf{A}^-\otimes\mathbf{a}^+ - q\mathbf{A}^-\otimes\mathbf{k}\otimes\mathbf{1}\otimes\mathbf{k} - q^2\mathbf{K}\otimes\mathbf{a}^-\otimes\mathbf{K}\otimes\mathbf{a}^+,\, \mathcal{K}] = 0,\\
\langle 34\rangle:\quad & (\mathbf{A}^-\otimes\mathbf{a}^+\otimes\mathbf{K}\otimes\mathbf{a}^- + \mathbf{K}\otimes\mathbf{a}^-\otimes\mathbf{A}^+\otimes\mathbf{a}^- - q\mathbf{K}\otimes\mathbf{k}\otimes\mathbf{1}\otimes\mathbf{k})\mathcal{K}\\
& = \mathcal{K}(\mathbf{A}^+\otimes\mathbf{a}^-\otimes\mathbf{K}\otimes\mathbf{a}^+ + \mathbf{K}\otimes\mathbf{a}^+\otimes\mathbf{A}^-\otimes\mathbf{a}^+ - q\mathbf{K}\otimes\mathbf{k}\otimes\mathbf{1}\otimes\mathbf{k}),\\
\langle 35\rangle:\quad & (\mathbf{A}^-\otimes\mathbf{a}^+\otimes\mathbf{K}\otimes\mathbf{k} + \mathbf{K}\otimes\mathbf{a}^-\otimes\mathbf{A}^+\otimes\mathbf{k} + \mathbf{K}\otimes\mathbf{k}\otimes\mathbf{1}\otimes\mathbf{a}^+)\mathcal{K} = F\mathcal{K}(\mathbf{1}\otimes\mathbf{k}\otimes\mathbf{K}\otimes\mathbf{a}^+),\\
\langle 42\rangle:\quad & (\mathbf{A}^+\otimes\mathbf{a}^-\otimes\mathbf{K}\otimes\mathbf{k} + \mathbf{K}\otimes\mathbf{a}^+\otimes\mathbf{A}^-\otimes\mathbf{k} + \mathbf{K}\otimes\mathbf{k}\otimes\mathbf{1}\otimes\mathbf{a}^-)\mathcal{K} = \mathcal{K}(\mathbf{1}\otimes\mathbf{k}\otimes\mathbf{K}\otimes\mathbf{a}^-),\\
\langle 43\rangle:\quad & (\mathbf{A}^+\otimes\mathbf{a}^-\otimes\mathbf{K}\otimes\mathbf{a}^+ + \mathbf{K}\otimes\mathbf{a}^+\otimes\mathbf{A}^-\otimes\mathbf{a}^+ - q\mathbf{K}\otimes\mathbf{k}\otimes\mathbf{1}\otimes\mathbf{k})\mathcal{K}\\
& = \mathcal{K}(\mathbf{A}^-\otimes\mathbf{a}^+\otimes\mathbf{K}\otimes\mathbf{a}^- + \mathbf{K}\otimes\mathbf{a}^-\otimes\mathbf{A}^+\otimes\mathbf{a}^- - q\mathbf{K}\otimes\mathbf{k}\otimes\mathbf{1}\otimes\mathbf{k}),\\
\langle 44\rangle:\quad & [\mathbf{A}^+\otimes\mathbf{a}^-\otimes\mathbf{A}^+\otimes\mathbf{a}^- - q\mathbf{A}^+\otimes\mathbf{k}\otimes\mathbf{1}\otimes\mathbf{k} - q^2\mathbf{K}\otimes\mathbf{a}^+\otimes\mathbf{K}\otimes\mathbf{a}^-,\, \mathcal{K}] = 0,\\
\langle 45\rangle:\quad & (\mathbf{A}^+\otimes\mathbf{a}^-\otimes\mathbf{A}^+\otimes\mathbf{k} + \mathbf{A}^+\otimes\mathbf{k}\otimes\mathbf{1}\otimes\mathbf{a}^+ - q^2\mathbf{K}\otimes\mathbf{a}^+\otimes\mathbf{K}\otimes\mathbf{k})\mathcal{K}\\
& = \mathcal{K}(\mathbf{1}\otimes\mathbf{a}^+\otimes\mathbf{1}\otimes\mathbf{k} + \mathbf{1}\otimes\mathbf{k}\otimes\mathbf{A}^+\otimes\mathbf{a}^-),\\
\langle 52\rangle:\quad & [\mathbf{1}\otimes\mathbf{k}\otimes\mathbf{K}\otimes\mathbf{k},\, \mathcal{K}] = 0 \quad (\text{same as } \langle 25\rangle),\\
\langle 53\rangle:\quad & (\mathbf{1}\otimes\mathbf{k}\otimes\mathbf{K}\otimes\mathbf{a}^+)\mathcal{K} = \mathcal{K}(\mathbf{A}^-\otimes\mathbf{a}^+\otimes\mathbf{K}\otimes\mathbf{k} + \mathbf{K}\otimes\mathbf{a}^-\otimes\mathbf{A}^+\otimes\mathbf{k} + \mathbf{K}\otimes\mathbf{k}\otimes\mathbf{1}\otimes\mathbf{a}^+),\\
\langle 54\rangle:\quad & (\mathbf{1}\otimes\mathbf{a}^+\otimes\mathbf{1}\otimes\mathbf{k} + \mathbf{1}\otimes\mathbf{k}\otimes\mathbf{A}^+\otimes\mathbf{a}^-)\mathcal{K}\\
& = \mathcal{K}(\mathbf{A}^+\otimes\mathbf{a}^-\otimes\mathbf{A}^+\otimes\mathbf{k} + \mathbf{A}^+\otimes\mathbf{k}\otimes\mathbf{1}\otimes\mathbf{a}^+ - q^2\mathbf{K}\otimes\mathbf{a}^+\otimes\mathbf{K}\otimes\mathbf{k}),\\
\langle 55\rangle:\quad & [\mathbf{1}\otimes\mathbf{a}^+\otimes\mathbf{1}\otimes\mathbf{a}^+ - q\mathbf{1}\otimes\mathbf{k}\otimes\mathbf{A}^+\otimes\mathbf{k},\, \mathcal{K}] = 0.
\end{aligned}
$$

For example (B.1) is obtained by taking the matrix element of $\langle 55\rangle$ for the transition $|a\rangle\otimes|i\rangle\otimes|b-1\rangle\otimes|j\rangle \to |c\rangle\otimes|m\rangle\otimes|d\rangle\otimes|n\rangle$.

The equations $\langle rs\rangle$ and $\langle sr\rangle$ are transformed into each other by $\mathcal{K} \leftrightarrow \mathcal{K}^{-1}$. Combining (3.24) and (3.25) with the argument similar to Proposition 2.4, one can also show that $\langle rs\rangle$

and $\langle s'r'\rangle$ are transformed into each other via the simultaneous interchange $(\mathbf{a}^+, \mathbf{A}^+, \mathcal{K}) \leftrightarrow (\mathbf{a}^-, \mathbf{A}^-, \mathcal{K}^{-1})$, where $r' = 7 - r$.

Appendix B. Proof of Theorem 3.4

Let $\langle rs\rangle$ be the intertwining relation for the matrix elements $\mathcal{K}^{cmdn}_{a\,i\,b\,j}$ as explained in Appendix A. The equations $\langle 25\rangle$ and $\langle 52\rangle$ give $(q^{2b+i+j} - q^{2d+m+n})\mathcal{K}^{cmdn}_{a\,i\,b\,j} = 0$. Note also that $\mathcal{K}^{cmdn}_{a\,i\,b\,j} = 0$ unless $a, b, c, d, i, j, m, n \geq 0$. Applying these properties and the normalization $\mathcal{K}^{0000}_{0000} = 1$ to the rest of $\langle rs\rangle$, one can deduce the conservation law implied by the factor $\delta_{c+m+d,a+i+b}\delta_{d+n-c,b+j-a}$ in (3.27).

First we reduce $\mathcal{K}^{cmdn}_{a\,i\,b\,j}$ to $b = 0$ case by means of $\langle 55\rangle$:

$$\mathcal{K}^{cmdn}_{a\,i\,b\,j} = q^{-i-j-1}\Big(-\mathcal{K}^{c,m-1,d,n-1}_{a,i,b-1,j} + \mathcal{K}^{c,m,d,n}_{a,i+1,b-1,j+1} + q^{m+n+1}\mathcal{K}^{c,m,d-1,n}_{a,i,b-1,j}\Big), \tag{B.1}$$

which is valid for $b \geq 1$ and $a, c, d, m, n, i, j \geq 0$. This can be fitted to a recursion relation of q^2-trinomial coefficients. The solution reads

$$\begin{aligned}\mathcal{K}^{cmdn}_{a\,i\,b\,j} =&\delta_{c+m+d,a+i+b}\delta_{d+n-c,b+j-a}\sum_{\alpha,\beta}\begin{Bmatrix} b \\ \alpha,\beta,b-\alpha-\beta\end{Bmatrix}(-1)^\alpha \\ &\times q^{(\alpha+\beta-b)(\alpha+\beta+b-1)+(m+n-2\alpha+1)\beta-(i+j+1)b}\mathcal{K}^{c,m-\alpha,d-\beta,n-\alpha}_{a,i+b-\alpha-\beta,0,j+b-\alpha-\beta},\end{aligned} \tag{B.2}$$

where the sum is over $\alpha, \beta \in \mathbb{Z}_{\geq 0}$, which is actually finite.

Second we reduce d by means of $\langle 22\rangle|_{b=0}$:

$$\mathcal{K}^{cmdn}_{a\,i\,0\,j} = \frac{q^{-m-n-1}}{1-q^{4d}}\Big(-(1-q^{2i})(1-q^{2j})\mathcal{K}^{c,m,d-1,n}_{a,i-1,0,j-1} + (1-q^{2m+2})(1-q^{2n+2})\mathcal{K}^{c,m+1,d-1,n+1}_{a,i,0,j}\Big),$$

which is valid for $d \geq 1$. By considering the combination $\frac{(q^4)_d(q^2)_m(q^2)_n}{(q^2)_i(q^2)_j}\mathcal{K}^{cmdn}_{a\,i\,0\,j}$, this is fitted with a recursion relation of q^2-binomial coefficients. The solution reads

$$\begin{aligned}\mathcal{K}^{cmdn}_{a\,i\,0\,j} = \delta_{c+m+d,a+i}\delta_{d+n-c,j-a}&\frac{q^{-(m+n+1)d}}{(q^4)_d}\begin{Bmatrix} i,j \\ m,n\end{Bmatrix}\sum_\gamma\begin{Bmatrix} d, m+d-\gamma, n+d-\gamma \\ \gamma, d-\gamma, i-\gamma, j-\gamma\end{Bmatrix}(-1)^\gamma \\ &\times q^{(\gamma-d)(\gamma+d-1)}\mathcal{K}^{c,m+d-\gamma,0,n+d-\gamma}_{a,i-\gamma,0,j-\gamma},\end{aligned} \tag{B.3}$$

where the sum over $\gamma \in \mathbb{Z}_{\geq 0}$ is finite. Combining (B.2) and (B.3), we obtain

$$\begin{aligned}\mathcal{K}^{cmdn}_{a\,i\,b\,j} &= \delta_{c+m+d,a+i+b}\delta_{d+n-c,b+j-a}\sum_{\alpha,\beta,\gamma}\frac{(-1)^{\alpha+\gamma}}{(q^4)_{d-\beta}}q^{\phi_1}\mathcal{K}^{c,m+d-\alpha-\beta-\gamma,0,n+d-\alpha-\beta-\gamma}_{a,i+b-\alpha-\beta-\gamma,0,j+b-\alpha-\beta-\gamma} \\ &\times\begin{Bmatrix} b, d-\beta, i+b-\alpha-\beta, j+b-\alpha-\beta, m+d-\alpha-\beta-\gamma, n+d-\alpha-\beta-\gamma \\ \alpha,\beta,\gamma,m-\alpha,n-\alpha,b-\alpha-\beta,d-\beta-\gamma,i+b-\alpha-\beta-\gamma,j+b-\alpha-\beta-\gamma\end{Bmatrix},\end{aligned} \tag{B.4}$$

where ϕ_1 is given in (3.27).

Next we reduce n and j in $\mathcal{K}^{cm0n}_{a\,i\,0\,j}$ to 0 keeping $d = b = 0$. Such recursion relations are available from $\langle 24\rangle|_{b=d=0}$ and $\langle 35\rangle|_{b=d=0}$:

$$\mathcal{K}^{cm0n}_{a\,i\,0\,j} = \frac{1}{1-q^{2n}}\left(q^{2a+i-m}(1-q^{2j})\mathcal{K}^{c,m,0,n-1}_{a,i,0,j-1} + q^{j-m}(1-q^{2i})\mathcal{K}^{c,m,0,n-1}_{a+1,i-1,0,j}\right), \tag{B.5}$$

$$\mathcal{K}^{cm0n}_{a\,i\,0\,j} = q^{2c-i+m}\mathcal{K}^{c,m,0,n-1}_{a,i,0,j-1} + q^{n-i}(1-q^{4c+4})\mathcal{K}^{c+1,m-1,0,n}_{a,i,0,j-1}, \tag{B.6}$$

which hold for $n \geq 1$ and $j \geq 1$, respectively. Note that either $(\mathrm{B.5})|_{j=0}$ or $(\mathrm{B.6})|_{n=0}$ leads to

$$\mathcal{K}^{ai00}_{ai00} = (-1)^i q^{2(a+1)i}$$

with the help of the conservation law (3.29) and $\mathscr{K}^{0000}_{0000} = 1$. It is easy to solve (B.5) and (B.6) with the above initial condition. The solution is given by (3.28). It fulfills the symmetry

$$\mathscr{K}^{cm0n}_{a\,i\,0\,j} = \frac{(q^4)_a}{(q^4)_c} \begin{Bmatrix} i,j \\ m,n \end{Bmatrix} \mathscr{K}^{a\,i\,0\,j}_{cm0n} \tag{B.7}$$

in accordance with (3.25). This is seen by replacing λ with $a - c + \lambda$ in (3.28). Finally (3.27) is obtained by applying (B.7) to (B.4).

Appendix C. Outline of the proof of Theorem 3.5

Let $\langle rs \rangle$ be the equation for $\mathscr{K}^{cmdn}_{a\,i\,b\,j}$ as in Appendix B. We are going to prove Theorem 3.5 solely by using the $\langle rs \rangle$'s, i.e. the characterization of $\mathscr{K}$ without relying on the explicit formula in Theorem 3.4.

Proof of claim (i) in Theorem 3.5. From Theorem 3.4, $\mathscr{K}^{cmdn}_{a\,i\,b\,j}$ is a rational function of q. We divide the proof into 2 Steps.

Step 1. We show $\mathscr{K}^{cmdn}_{a\,i\,b\,j} \in \mathbb{Z}[q, q^{-1}]$. First we see that (B.1) attributes the claim to $b = 0$ case $\mathscr{K}^{cmdn}_{a\,i\,0\,j}$. One can utilize similar relations $\langle 35 \rangle|_{b=0}$, $\langle 45 \rangle|_{b=j=0}$ and $\langle 44 \rangle|_{b=j=i=0}$ with coefficients in $\mathbb{Z}[q, q^{-1}]$ to reduce the indices j, i, a successively and thereby the claim itself to $\mathscr{K}^{cmdn}_{0\,0\,0\,0}$. But the last quantity is $\delta_{c0}\delta_{m0}\delta_{d0}\delta_{n0}$ by the conservation law and we are done.

Step 2. We show $\mathscr{K}^{cmdn}_{a\,i\,b\,j} \in A$, where A is the ring of rational functions of q regular at $q = 0$. (Plainly, $\lim_{q\to 0} \mathscr{K}^{cmdn}_{a\,i\,b\,j} < \infty$.) Our strategy is similar to Step 1 but this time with coefficients from A instead of $\mathbb{Z}[q, q^{-1}]$. First solving (B.1) for $\mathscr{K}^{c,m,d,n}_{a,i+1,b-1,j+1}$ we see that the claim $\mathscr{K}^{cmdn}_{a\,i\,b\,j} \in A$ for $i, j \geq 1$ is attributed to $\min(i, j) = 0$ case. Thus we consider the reductions of $\mathscr{K}^{cmdn}_{a\,i\,b\,0}$ and $\mathscr{K}^{cmdn}_{a\,0\,b\,j}$. For the former, $\langle 45 \rangle|_{j=0}$ provides the relation $\mathscr{K}^{cmdn}_{a\,i\,b\,0} = \sum A\mathscr{K}^{\bullet,\bullet,\bullet,\bullet}_{\bullet,i-1,\bullet,0}$. For the latter one eliminates $\mathscr{K}^{cmdn}_{a\,1\,b\,j}$ between $\langle 54 \rangle|_{i=0}$ and $\langle 55 \rangle|_{i=0}$ to get a relation $\mathscr{K}^{cmdn}_{a\,0\,b\,j} = \sum A\mathscr{K}^{\bullet,\bullet,\bullet,\bullet}_{\bullet,0,\bullet,j-1} + \sum A\mathscr{K}^{\bullet,\bullet,\bullet,\bullet}_{\bullet,0,\bullet,j-2}$. Thus the claim is reduced to $\mathscr{K}^{cmdn}_{a\,0\,b\,0}$ in the both cases. Next we utilize $\langle 44 \rangle|_{i=j=0}$ having the form $\mathscr{K}^{cmdn}_{a\,0\,b\,0} = \sum A\mathscr{K}^{\bullet,m-1,\bullet,\bullet}_{\bullet,0,\bullet,0} + \sum A\mathscr{K}^{\bullet,m-2,\bullet,\bullet}_{\bullet,0,\bullet,0}$. Thus the claim is reduced to $\mathscr{K}^{c\,0\,d\,n}_{a\,0\,b\,0}$. Now we eliminate $\mathscr{K}^{c\,1\,d\,n}_{a\,0\,b\,0}$ between $\langle 22 \rangle|_{i=j=m=0}$ and $\langle 44 \rangle|_{i=j=m=0}$ to get $\mathscr{K}^{c\,0\,d\,n}_{a\,0\,b\,0} = \sum A\mathscr{K}^{\bullet,0,\bullet,\bullet}_{a-1,0,\bullet,0}$ reducing the claim further to $\mathscr{K}^{c\,0\,d\,n}_{0\,0\,b\,0}$. Similarly eliminate $\mathscr{K}^{c\,1\,d\,n}_{0\,0\,b\,0}$ between $\langle 22 \rangle|_{i=j=m=a=0}$ and $\langle 23 \rangle|_{i=j=m=a=0}$ to get $\mathscr{K}^{c\,0\,d\,n}_{0\,0\,b\,0} = \sum A\mathscr{K}^{\bullet,0,d-1,\bullet}_{0,0,\bullet,0}$ reducing the claim down to $\mathscr{K}^{c\,0\,0\,n}_{0\,0\,b\,0} = \delta_{c,b}\delta_{n,2b}\mathscr{K}^{b\,0\,0\,2b}_{0\,0\,b\,0}$, where the last equality is due to the conservation law (3.29). Finally we eliminate $\mathscr{K}^{c\,0\,0\,n}_{0\,1\,b\,0}$ between $\langle 24 \rangle|_{i=j=m=a=d=0}$ and $\langle 45 \rangle|_{i=j=m=a=d=0}$ to find $\mathscr{K}^{b\,0\,0\,2b}_{0\,0\,b\,0} = (b \to b-1) = \cdots = \mathscr{K}^{0000}_{0000} = 1$. The claim (i) has been proved.

Proof of claim (ii) in Theorem 3.5. By the definition (3.30) of $\mathcal{K}$ and (3.25) we a priori know $\mathcal{K}^{cmdn}_{a\,i\,b\,j} = \mathcal{K}^{a\,i\,b\,j}_{cmdn}$. The claim (ii) must be consistent with this property. It postulates that the map $(c, m, d, n) \mapsto (c', m', d', n')$ defined by the piecewise linear formula (3.31) must be involutive. It is certainly so because the formula arises as the ultradiscretization of $\mathbf{K}$ and $\mathbf{K} = \mathbf{K}^{-1}$ holds. See the comment after (3.41).

By the argument so far, the formula (3.31) for $\mathcal{K}^{cmdn}_{a\,i\,b\,j}$ is equivalent to that obtained under the interchange $(c, m, d, n) \leftrightarrow (a, i, b, j)$. This observation halves our task in what follows. From (3.29) we will always take it for granted that $\mathcal{K}^{cmdn}_{a\,i\,b\,j}$ also obeys the same conservation law.

Now we begin reducing the indices as in the proof of the claim (i) by using $\langle rs \rangle|_{q=0}$. Consider for example (B.1), multiply it with q^{i+j+1} and set $q = 0$. Now that we know $\mathscr{K}$'s are polynomials in q, the result gives $\mathcal{K}^{c,m,d,n}_{a,i+1,b-1,j+1} = \mathcal{K}^{c,m-1,d,n-1}_{a,\,i,\,b-1,\,j}$ for the constant terms. Similarly $\langle 33 \rangle|_{q=0}$ yields $\mathcal{K}^{c,m,d,n}_{a-1,i+1,b-1,j+1} = \mathcal{K}^{c+1,m-1,d+1,n-1}_{a,\,i,\,b,\,j}$. One can check that the piecewise linear formula (3.31) also satisfies these recursion relations. Thus a proof for $\mathcal{K}^{cmdn}_{a\,i\,b\,j}$ is attributed to the situation $\min(a, b, c, d) = \min(m, n, i, j) = 0$. There are $4 \times 4 = 16$ such possibilities but the symmetry $(a, m, d, n) \leftrightarrow (a, b, i, j)$ already established in the above halves them to the 8 cases

as follows. (Their choice is not unique.)

$$\text{(I) } d = n = 0, \quad \text{(II) } c = n = 0, \quad \text{(III) } b = n = 0, \quad \text{(IV) } a = n = 0,$$
$$\text{(V) } d = m = 0, \quad \text{(VI) } c = m = 0, \quad \text{(VII) } b = m = 0, \quad \text{(VIII) } a = m = 0.$$

We have proved them all case by case. Here we illustrate the proof for (I) only. The other cases can be treated similarly.

For (I), the formula (3.31) to be proved reads $\mathcal{K}^{cm00}_{a\,i\,b\,j} = \delta_{a,c+m}\delta_{i,0}\delta_{b,0}\delta_{j,m}$. First note that $(\mathrm{B}.1)|_{d=n=0}$ in the limit $q \to 0$ tells that $\mathcal{K}^{cm00}_{a\,i\,b\,j} = 0$ if $\min(i,j) \geq 1$. This agrees with the formula to be shown in the region $\min(i,j) \geq 1$. Thus we are left to verify (I-1) $\mathcal{K}^{cm00}_{a\,0\,b\,j} = \delta_{a,c+m}\delta_{b,0}\delta_{j,m}$ and (I-2) $\mathcal{K}^{cm00}_{a\,i\,b\,0} = \delta_{a,c}\delta_{i,0}\delta_{b,0}\delta_{0,m}$.

To prove (I-1), we utilize $\langle 54\rangle|_{d=n=i=0}$ in the limit $q \to 0$, which says $\mathcal{K}^{cm00}_{a\,0\,b\,j} = \mathcal{K}^{c,m-1,0,0}_{a-1,0,b,j-1}$. Note that the RHS of (I-1) is also invariant under the simultaneous decrement of a, m, j by 1. Since the conservation law implies $a, m \geq j$ for nonzero $\mathcal{K}^{cm00}_{a\,0\,b\,j}$, the above recursion reduces (I-1) to $j = 0$ case, namely $\mathcal{K}^{cm00}_{a\,0\,b\,0} = \delta_{a,c}\delta_{b,0}\delta_{m,0}$. By the conservation law, the LHS equals $\delta_{a,b+c}\delta_{m,2b}\mathcal{K}^{c,2b,0,0}_{b+c,0,b,0}$. Thus we are to show $\mathcal{K}^{c,2b,0,0}_{b+c,0,b,0} = \delta_{b,0}$. From $\langle 42\rangle|_{d=n=i=j=0}$ in the limit $q \to 0$, we find that $\mathcal{K}^{c,2b,0,0}_{b+c,0,b,0}$ indeed vanishes for $b \geq 1$. It remains to check $\mathcal{K}^{c000}_{c000} = 1$. From $\langle 44\rangle|_{d=n=i=j=m=b=0}$ (with q generic), we find $\mathscr{K}^{c000}_{c000} = (c \to c-1) = \cdots = \mathscr{K}^{0000}_{0000} = 1$, completing the proof of (I-1).

To prove (I-2), we utilize $\langle 23\rangle|_{d=n=j=0}$ in the limit $q \to 0$, which says $\mathcal{K}^{cm00}_{a\,i\,b\,0} = \mathcal{K}^{c,m-1,0,0}_{a-1,i+1,b-1,0}$. Since the conservation law implies $a, m \geq b$ for nonzero $\mathcal{K}^{cm00}_{a\,i\,b\,0}$, the above recursion reduces the LHS of (I-2) to $b = 0$ case. Thus we are to verify $\mathcal{K}^{c,m-b,0,0}_{a-b,i+b,0,0} = \delta_{a,c}\delta_{i,0}\delta_{b,0}\delta_{0,m}$. Since the conservation law limits the nontrivial case to $a = b + c$ and $m = i + 2b$, it boils down to showing $\mathcal{K}^{\alpha\beta00}_{\alpha\beta00} = \delta_{\beta,0}$. The vanishing for $\beta \geq 1$ follows from $\langle 23\rangle|_{d=n=j=b=0}$ in the limit $q \to 0$, and $\beta = 0$ case has already been shown in the end of the proof of (I-1). We have finished the proof of (I-2) and thereby (I).

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